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Asymptotic theory of the Boltzmann system, for a steady flow of a slightly rarefied gas with a finite Mach number: General theory

Yoshio Sone [a,*], Claude Bardos [b], François Golse [c], Hiroshi Sugimoto [a]

[a] *Department of Aeronautics and Astronautics, Graduate School of Engineering, Kyoto University, Kyoto 606-8501, Japan*
[b] *Départment de Mathématiques, Unversité Paris VII; CLMA, Ecole Normale Supérieure Cachan, 61 Avenue du Président Wilson, 94235 Cachan, France*
[c] *Ecole Normale Supérieure, DMI, 45 rue d'Ulm, 75005 Paris, France*

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Abstract – A steady rarefied gas flow with Mach number of the order of unity around a body or bodies is considered. The general behaviour of the gas for small Knudsen numbers is studied by asymptotic analysis of the boundary-value problem of the Boltzmann equation for a general domain. The effect of gas rarefaction (or Knudsen number) is expressed as a power series of the square root of the Knudsen number of the system. A series of fluid-dynamic type equations and their associated boundary conditions that determine the component functions of the expansion of the density, flow velocity, and temperature of the gas is obtained by the analysis. The equations up to the order of the square root of the Knudsen number do not contain non-Navier–Stokes stress and heat flow, which differs from the claim by Darrozes (in Rarefied Gas Dynamics, Academic Press, New York, 1969). The contributions up to this order, except in the Knudsen layer, are included in the system of the Navier–Stokes equations and the slip boundary conditions consisting of tangential velocity slip due to the shear of flow and temperature jump due to the temperature gradient normal to the boundary. © 2000 Éditions scientifiques et médicales Elsevier SAS

rarefied gas / Boltzmann equation / slip flow / kinetic theory / boundary layer

1. Introduction

The study of the relation of the two systems describing the behaviour of a gas, the system of classical gas dynamics and the Boltzmann system, has a long history (Boltzmann [1], Hilbert [2], Chapman [3], Enskog [4], Grad [5], Darrozes [6], Sone [7–9], Sone et al. [10], etc.). The first mathematical analysis is the work of Hilbert, where the solution of the Boltzmann equation is obtained by expanding the velocity distribution function and the macroscopic variables, such as density, flow velocity, and temperature, in a power-series of the Knudsen number. In this solution, the velocity distribution function is expressed in terms of the component functions of the expansion of the macroscopic variables: density, flow velocity, and temperature (or pressure), and a series of equations for these component functions of the macroscopic variables is given. (The set of equations of macroscopic variables that determines the behaviour of a gas will be called fluid-dynamic type equations.) The set of equations appearing at the leading order of the expansion is the Euler set of equations for the leading terms of the expansion of the macroscopic variables. The fluid-dynamic type equations in the following orders consist of two parts: (i) the terms that are derived from the Euler set of equations by substituting the expansions of the macroscopic variables in the Euler set of equations and arranging the same order terms of the Knudsen number and (ii) the inhomogeneous terms that enter as the contribution of stress and heat flow expressed by derivatives of lower-order quantities. In this system the Navier–Stokes set of equations never appears. (The

* Correspondence and reprints; e-mail: sone@suma.kuaero.kyoto-u.ac.jp

reason is made clear in Sone [8]; see also Sone and Aoki [11] and Sone [12].) Probably because of this reason, Chapman and Enskog, independently, developed a skillful expansion, where the velocity distribution function is expanded, but the macroscopic variables are not, and obtained a series of fluid-dynamic type equations. The leading set of equations is the Euler set of equations, the second is the Navier–Stokes set of equations, the third is the Burnett set of equations, and so on. Owing to the derivation of the Navier–Stokes set of equations, the Chapman–Enskog expansion is mainly referred to when the relation between fluid dynamics and kinetic theory is mentioned. However, disadvantages of this expansion are also reported. For example, a non-well-posed set of equations appears after the Navier–Stokes set (e.g., Sone [13]); it shows some unfavourable behaviour of solution when a boundary-value problem is considered (Cercignani [14]); the order of the differential of the equations increases as the approximation advances, which introduces problems for the construction of its boundary condition. Another point, which is not so well noticed, is that the Navier–Stokes set is just a perturbation of the Euler set. Thus there may naturally arise the question of whether the Navier–Stokes set is derived directly for the system of moderate Reynolds numbers. These kinds of questions can be made clear by the Hilbert expansion, without bypassing via the technical Chapman–Enskog expansion, if one specifies the situation that one wants to discuss, and if the correct ordering of the sizes of the variables is made beforehand. (See, for example, Sone and Aoki [11], Sone [12].) Incidentally, the structure of Chapman–Enskog expansion, for which the original articles are not clear-cut and are lengthy, is clearly and briefly explained in Grad [15].

In these two expansions themselves, an initial- or boundary-value problem is not taken into account. A general theory for initial problems is developed by Grad [5], where the initial layer is first introduced and the initial slip condition is analysed systematically. The boundary-value problem for the steady behaviour of a gas for small Knudsen numbers in a general domain is also discussed (e.g., Darrozes [6], Sone [7–9], Sone and Aoki [16], Sone et al. [10].) In Sone [7–9], the case where the state of the gas is close to a uniform state at rest is considered, and its asymptotic behaviour for small Knudsen numbers is analysed on the basis of the linearized Boltzmann equation. The overall behaviour of the gas is described by the Stokes set of equations at any order of approximation. The boundary condition for the Stokes set and the Knudsen-layer correction in the neighbourhood of the boundary are obtained up to the second order of the Knudsen number. At the second order of the Knudsen number, a thin layer with thickness of the order of the mean free path squared divided by the radius of the curvature of the boundary appears at the bottom of the Knudsen layer over a convex boundary (Sone [17]). This is due to the discontinuity of the velocity distribution function of the gas molecules over a boundary (Sone and Takata [18]). The slip boundary conditions up to this order, as well as the fluid-dynamic type equations, are not affected by the existence of this layer. Sone [8] and Sone and Aoki [16] extended the linearized theory, which corresponds to the case where the Reynolds number of the system is very small ($\mathrm{Re} \ll 1$), to the case where the Reynolds number is of the order of unity, for which the linearized Boltzmann equation is no longer valid, and the expansion is carried out by taking into account the relation among three important parameters: Mach number Ma is of the same order as the product of Reynolds number Re and the Knudsen number Kn ($\mathrm{Ma} \propto \mathrm{Re\,Kn}$). The leading fluid-dynamic type equations are the Navier–Stokes set of equations for incompressible fluids. The next-order equations are the second set of equations of the Mach number expansion of the Navier–Stokes set for compressible fluids with an additional thermal stress term in the momentum equation owing to a rarefaction effect of the gas. The boundary conditions for these sets are the nonslip condition for the incompressible Navier–Stokes set and the slip condition consisting of tangential velocity slip due to the shear of flow and the temperature gradient along the boundary and temperature jump due to the temperature gradient normal to the boundary for the next-order set. It should be noted that the slip condition alone for the Navier–Stokes set is not sufficient to obtain the correct first-order effect of gas rarefaction. The restriction imposed in this work that the variation of the temperature of the boundary be small (or the same order as the Knudsen number) is eliminated in Sone et al. [10], where an important result, which shows incompleteness of the continuum gas dynamics (classical gas dynamics), is derived. The above-mentioned works on the boundary-value problem are limited to small flow velocity (or a small Mach number). Darrozes [6] considered the case of a finite Mach number. He claimed that the behaviour of the gas is conveniently described by splitting the domain into three regions: the overall domain described by Euler-type equations, the viscous boundary layer in the neighbourhood of a boundary, with thickness of the order of the square root of the mean free path, and the Knudsen layer at the bottom of the viscous boundary layer with thickness of the order of the mean free path. One of his main conclusions is that the boundary-layer equations describing the leading effect of gas rarefaction contain terms that are not contained in the Navier–Stokes set for compressible fluids. Incidentally, when a finite Mach number is mentioned, one may think that this is a general case that covers a small Mach number case. This is not so in the case of small Knudsen numbers, because a finite Mach number means a very large Reynolds number owing to the above-mentioned relation among the three parameters. Thus, a finite Mach number is, rather, a special case to be treated separately.

Darrozes's work, though pioneering, is simple, and the systematic system of formulas, as well as a course of analysis, that is required to understand the theory and to apply it to concrete problems is not given. Thus we reconsider the problem and develop a complete asymptotic theory of the boundary-value problem of the Boltzmann equation for small Knudsen numbers, where the viscous boundary-layer equations are derived directly, not as a perturbation of the Euler equations. Some important results as well as a systematic set of formulas, which are not described in Darrozes [6] and even disagree with his main statement, will be shown in the following analysis.

2. Problem and basic equation

We consider a rarefied gas in a general domain except that the shape of the domain is smooth. The boundary of the domain is a solid boundary through which there is no mass flux, and the characteristic speed of gas flow compared with the sonic speed is of the order of unity (or the Mach number is of the order of unity). We will investigate the asymptotic behaviour of the gas when the mean free path of the gas molecules is small compared with the geometrical characteristic length of the system (or the Knudsen number of the system is small), and try to find the fluid-dynamic type equations and their associated boundary conditions. The analysis is carried out under the following assumptions: (i) the behaviour of the gas is described by the Boltzmann equation; (ii) the gas molecules make the diffuse reflection on the boundary. (The extension to a more general condition is noted in the course of analysis.) In the following analysis we use nondimensional variables characterized by the equilibrium state at rest with density ρ_0 and temperature T_0 (thus, pressure $p_0 = R\rho_0 T_0$; R: the specific gas constant or the Boltzmann constant k_B divided by the mass m of a molecule).

The Boltzmann equation for a steady state is written in the nondimensional form:

$$\zeta_i \frac{\partial \widehat{\Phi}}{\partial x_i} = \frac{1}{k} \widehat{J}(\widehat{\Phi}, \widehat{\Phi}), \tag{1}$$

where $(2RT_0)^{1/2}\zeta_i$ or $(2RT_0)^{1/2}\boldsymbol{\zeta}$ is the molecular velocity, $x_i = X_i/L$ (X_i: the space coordinates, L: a characteristic length of the system), $\rho_0(2RT_0)^{-3/2}\widehat{\Phi}$ is the velocity distribution function of the gas molecules, $k = \sqrt{\pi}\mathrm{Kn}/2 = \sqrt{\pi}l_0/2L$ (Kn: the Knudsen number; l_0: the mean free path of the gas molecules in the equilibrium state at rest with pressure p_0 and temperature T_0. For a hard-sphere molecular gas, $l_0 = m/\sqrt{2}\pi d_m^2 \rho_0$, where d_m is the diameter of a molecule.), and $\widehat{J}(\widehat{\Phi}, \widehat{\Phi})$ is the collision integral.

The collision integral is expressed as

$$\widehat{J}(\phi, \psi) = \frac{1}{2}\int \left(\phi'_*\psi' + \phi'\psi'_* - \phi_*\psi - \phi\psi_*\right) \widehat{B}\left(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|\right) \mathrm{d}\Omega(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\zeta}_*, \tag{2}$$

$$\phi=\phi(\boldsymbol{\zeta}),\quad \phi_*=\phi(\boldsymbol{\zeta}_*),\quad \phi'=\phi(\boldsymbol{\zeta}'),\quad \phi'_*=\phi(\boldsymbol{\zeta}'_*),\quad \text{etc.},$$
$$\boldsymbol{\zeta}'=\boldsymbol{\zeta}+\boldsymbol{\alpha}\big[\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_*-\boldsymbol{\zeta})\big],\quad \boldsymbol{\zeta}'_*=\boldsymbol{\zeta}_*-\boldsymbol{\alpha}\big[\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_*-\boldsymbol{\zeta})\big],\ \mathrm{d}\boldsymbol{\zeta}_*=\mathrm{d}\zeta_{1*}\,\mathrm{d}\zeta_{2*}\,\mathrm{d}\zeta_{3*},\tag{3}$$

where $\widehat{B}$ is a nonnegative function determined by the type of the intermolecular potential, $\boldsymbol{\alpha}$ is a unit vector, $\mathrm{d}\Omega(\boldsymbol{\alpha})$ is the solid angle element in the direction of $\boldsymbol{\alpha}$, $\boldsymbol{\zeta}_*$ is the variable of integration corresponding to $\boldsymbol{\zeta}$, and the integration is carried out over the whole space of $\boldsymbol{\alpha}$ and that of $\boldsymbol{\zeta}_*$. For a hard-sphere molecular gas, the function $\widehat{B}$ is given by

$$\widehat{B}=\big|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_*-\boldsymbol{\zeta})\big|/4(2\pi)^{1/2}.\tag{4}$$

Further it should be noted that $\widehat{B}$ for a gas with a general intermolecular potential depends on the parameter U_0/mRT_0, where U_0 is a characteristic magnitude of the intermolecular potential. (See Sone and Aoki [11]). In this paper this fact is simply to be remembered.

The diffuse reflection condition on the boundary (or at $x_i=x_{wi}$) is expressed as

$$\widehat{\Phi}=\frac{\hat{\sigma}_w}{(\pi\hat{\tau}_w)^{3/2}}\exp\left(-\frac{(\zeta_i-\hat{u}_{wi})^2}{\hat{\tau}_w}\right)\quad(\zeta_i n_i>0),\tag{5}$$

$$\hat{\sigma}_w=-2\sqrt{\frac{\pi}{\hat{\tau}_w}}\int_{\zeta_i n_i<0}\zeta_j n_j\widehat{\Phi}\,\mathrm{d}\boldsymbol{\zeta},\quad \mathrm{d}\boldsymbol{\zeta}=\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2\,\mathrm{d}\zeta_3,\tag{6}$$

where n_i is the direction cosine of the normal to the boundary, pointing to the gas region, $(2RT_0)^{1/2}\hat{u}_{wi}$ ($\hat{u}_{wi}n_i=0$) is the velocity of the boundary, and $T_0\hat{\tau}_w$ is the temperature of the boundary. The condition $\hat{u}_{wi}n_i=0$ is required for the system to be steady.

The macroscopic variables of the gas, the density ρ or $\rho_0\hat{\omega}$, the flow velocity v_i or $(2RT_0)^{1/2}\hat{u}_i$, the temperature T or $T_0\hat{\tau}$, the pressure p or $p_0\hat{p}$, the stress tensor p_{ij} or $p_0\widehat{P}_{ij}$, and the heat-flow vector q_i or $p_0(2RT_0)^{1/2}\widehat{Q}_i$, are expressed by $\widehat{\Phi}$:

$$\hat{\omega}=\int\widehat{\Phi}\,\mathrm{d}\boldsymbol{\zeta},\tag{7a}$$

$$\hat{\omega}\hat{u}_i=\int\zeta_i\widehat{\Phi}\,\mathrm{d}\boldsymbol{\zeta},\tag{7b}$$

$$\frac{3}{2}\hat{\omega}\hat{\tau}=\int(\zeta_j-\hat{u}_j)^2\widehat{\Phi}\,\mathrm{d}\boldsymbol{\zeta},\tag{7c}$$

$$\hat{p}=\hat{\omega}\hat{\tau},\tag{7d}$$

$$\widehat{P}_{ij}=2\int(\zeta_i-\hat{u}_i)(\zeta_j-\hat{u}_j)\widehat{\Phi}\,\mathrm{d}\boldsymbol{\zeta},\tag{7e}$$

$$\widehat{Q}_i=\int(\zeta_i-\hat{u}_i)(\zeta_j-\hat{u}_j)^2\widehat{\Phi}\,\mathrm{d}\boldsymbol{\zeta},\tag{7f}$$

where the integration, and in what follows unless otherwise stated, is carried out over the whole space of $\boldsymbol{\zeta}$.

For the Boltzmann–Krook–Welander model or BKW model (Kogan [19]), the collision integral $\widehat{J}(\widehat{\Phi},\widehat{\Phi})$ is expressed as

$$\widehat{J}(\widehat{\Phi},\widehat{\Phi})=\hat{\omega}\big(\widehat{\Phi}_e-\widehat{\Phi}\big),\tag{8}$$

where $\widehat{\Phi}_e$ is the local Maxwellian corresponding to $\widehat{\Phi}$:

$$\widehat{\Phi}_e=\frac{\hat{\omega}}{(\pi\hat{\tau})^{3/2}}\exp\left(-\frac{(\zeta_i-\hat{u}_i)^2}{\hat{\tau}}\right).\tag{9}$$

We will investigate the asymptotic behaviour for small k of the solution of the boundary-value problem of equation (1) with equation (5) in a general domain.

3. Asymptotic analysis

3.1. Preliminary

Our previous asymptotic analyses for small Knudsen numbers of the boundary-value problems of the Boltzmann equation for a general domain are carried out in the following way. First a solution whose length scale of variation is the characteristic length of the system is obtained by a modified Hilbert expansion. The adjective 'modified' is attached, since in the analyses, first the situation to be considered being classified clearly and the size of the parameters characterizing the system being determined, then the relative size of the Knudsen number and the parameters is taken into account in the analysis (or expansion). The Hilbert solution is determined by the behaviour of the macroscopic variables, density, flow velocity, and temperature (or pressure). These variables are governed by a set of equations called fluid-dynamic type equations. The solution thus obtained does not generally satisfy the boundary condition on the boundary (on the level of the velocity distribution function). Thus, a correction (Knudsen-layer correction) is introduced. From analysis of the Knudsen layer, the boundary condition (or so called slip condition) for the fluid-dynamic type equations as well as the Knudsen-layer correction is obtained. We will generally follow this procedure in the following analysis.

The case with a finite Mach number and a small Knudsen number that we are going to investigate corresponds to a case with a very large Reynolds number, seen from the general relation among these three parameters:

$$\mathrm{Ma} \propto \mathrm{Re}\,\mathrm{Kn}. \tag{10}$$

(This relation is sometimes called von Karman relation but its exact relation including numerical factor has had to wait for a rigorous analysis such as the present one.) According to the analysis of the Navier–Stokes set of equations for a high Reynolds-number flow around a body, the flow field is split into two regions of different characters: the Prandtl boundary layer in the neighbourhood of the boundary, where the viscous effect is important, and the region outside the layer where the behaviour of the fluid is determined by the Euler set of equations. In view of this behaviour of solutions of the Navier–Stokes set of equations and with the expectation of the resemblance of the behaviour of a gas of small Knudsen numbers to that of the Navier–Stokes system, an intermediate region will be introduced between the two regions: the overall region and the Knudsen layer.

3.2. Hilbert solution

In this subsection, we consider the case where an appreciable variation of the state of gas occurs in the distance of the characteristic length L of the system (or where the length scale of variation of the variables is of the order of L), i.e. $\partial\widehat{\Phi}/\partial x_i = \mathrm{O}(\widehat{\Phi})$. The asymptotic solution for small k describing this situation is the well-known Hilbert solution (Hilbert [2]), which is obtained by a power series expansion in k under the assumption $\partial\widehat{\Phi}/\partial x_i = \mathrm{O}(\widehat{\Phi})$. In the analysis of the following subsection, where the adjustment to the boundary condition is considered, we encounter a series expansion in the square root of k, and therefore we will rewrite the Hilbert solution in a power series of $\sqrt{k}$. Here we introduce the notation ε:

$$\varepsilon = \sqrt{k}. \tag{11}$$

The Hilbert solution $\widehat{\Phi}_H$ is now rewritten as follows:

$$\widehat{\Phi}_H = \widehat{\Phi}_{h0} + \widehat{\Phi}_{h1}\varepsilon + \cdots, \tag{12}$$

where the subscript h is chosen to discriminate the components of the present expansion from those of the original Hilbert expansion. Correspondingly, the macroscopic variables $\hat{\omega}_H$, $\hat{u}_{iH}$, $\hat{\tau}_H$, $\hat{p}_H$, $\widehat{P}_{ijH}$, and $\widehat{Q}_{iH}$ defined by equations (7a)–(7f) with $\widehat{\Phi} = \widehat{\Phi}_H$ are also expanded as

$$\hat{h}_H = \hat{h}_{h0} + \hat{h}_{h1}\varepsilon + \cdots, \tag{13}$$

where $\hat{h}_H$ represents any of the macroscopic variables $\hat{\omega}_H$, $\hat{u}_{iH}$, $\hat{\tau}_H$, etc. By substitution of equation (12) into equations (7a)–(7f), the component function $\hat{h}_{hm}$ of $\hat{h}_H$ is expressed in terms of moments of $\widehat{\Phi}_{hn}$ $(n \leqslant m)$ (note the nonlinearity of equations (7b)–(7f)):

$$\hat{\omega}_{h0} = \int \widehat{\Phi}_{h0}\,\mathrm{d}\boldsymbol{\zeta}, \tag{14a}$$

$$\hat{\omega}_{h0}\hat{u}_{ih0} = \int \zeta_i \widehat{\Phi}_{h0}\,\mathrm{d}\boldsymbol{\zeta}, \tag{14b}$$

$$\frac{3}{2}\hat{\omega}_{h0}\hat{\tau}_{h0} = \int (\zeta_j - \hat{u}_{jh0})^2 \widehat{\Phi}_{h0}\,\mathrm{d}\boldsymbol{\zeta}, \tag{14c}$$

$$\hat{p}_{h0} = \hat{\omega}_{h0}\hat{\tau}_{h0}, \tag{14d}$$

$$\widehat{P}_{ijh0} = 2\int (\zeta_i - \hat{u}_{ih0})(\zeta_j - \hat{u}_{jh0}) \widehat{\Phi}_{h0}\,\mathrm{d}\boldsymbol{\zeta}, \tag{14e}$$

$$\widehat{Q}_{ih0} = \int (\zeta_i - \hat{u}_{ih0})(\zeta_j - \hat{u}_{jh0})^2 \widehat{\Phi}_{h0}\,\mathrm{d}\boldsymbol{\zeta}, \tag{14f}$$

$$\hat{\omega}_{h1} = \int \widehat{\Phi}_{h1}\,\mathrm{d}\boldsymbol{\zeta}, \tag{15a}$$

$$\hat{\omega}_{h0}\hat{u}_{ih1} = \int (\zeta_i - \hat{u}_{ih0}) \widehat{\Phi}_{h1}\,\mathrm{d}\boldsymbol{\zeta}, \tag{15b}$$

$$\frac{3}{2}\hat{\omega}_{h0}\hat{\tau}_{h1} = \int \left[(\zeta_j - \hat{u}_{jh0})^2 - \frac{3}{2}\hat{\tau}_{h0}\right] \widehat{\Phi}_{h1}\,\mathrm{d}\boldsymbol{\zeta}, \tag{15c}$$

$$\hat{p}_{h1} = \hat{\omega}_{h1}\hat{\tau}_{h0} + \hat{\omega}_{h0}\hat{\tau}_{h1}, \tag{15d}$$

$$\widehat{P}_{ijh1} = 2\int (\zeta_i - \hat{u}_{ih0})(\zeta_j - \hat{u}_{jh0}) \widehat{\Phi}_{h1}\,\mathrm{d}\boldsymbol{\zeta}, \tag{15e}$$

$$\widehat{Q}_{ih1} = \int (\zeta_i - \hat{u}_{ih0})(\zeta_j - \hat{u}_{jh0})^2 \widehat{\Phi}_{h1}\,\mathrm{d}\boldsymbol{\zeta} - \frac{3}{2}\hat{p}_{h0}\hat{u}_{ih1} - \widehat{P}_{ijh0}\hat{u}_{jh1}, \tag{15f}$$

$$\hat{\omega}_{h2} = \int \widehat{\Phi}_{h2}\,\mathrm{d}\boldsymbol{\zeta}, \tag{16a}$$

$$\hat{\omega}_{h0}\hat{u}_{ih2} = \int (\zeta_i - \hat{u}_{ih0}) \widehat{\Phi}_{h2}\,\mathrm{d}\boldsymbol{\zeta} - \hat{\omega}_{h1}\hat{u}_{ih1}, \tag{16b}$$

$$\frac{3}{2}\hat{\omega}_{h0}\hat{\tau}_{h2} = \int \left[(\zeta_j - \hat{u}_{jh0})^2 - \frac{3}{2}\hat{\tau}_{h0}\right] \widehat{\Phi}_{h2}\,\mathrm{d}\boldsymbol{\zeta} - \frac{3}{2}\hat{\omega}_{h1}\hat{\tau}_{h1} - 2\hat{\omega}_{h0}\hat{u}_{ih1}\hat{u}_{ih1}, \tag{16c}$$

$$\hat{p}_{h2} = \hat{\omega}_{h2}\hat{\tau}_{h0} + \hat{\omega}_{h1}\hat{\tau}_{h1} + \hat{\omega}_{h0}\hat{\tau}_{h2}, \tag{16d}$$

$$\widehat{P}_{ijh2} = 2\int (\zeta_i - \hat{u}_{ih0})(\zeta_j - \hat{u}_{jh0})\widehat{\Phi}_{h2}\,\mathrm{d}\boldsymbol{\zeta} - 2\hat{\omega}_{h0}\hat{u}_{ih1}\hat{u}_{jh1}, \tag{16e}$$

$$\widehat{Q}_{ih2} = \int (\zeta_i - \hat{u}_{ih0})(\zeta_j - \hat{u}_{jh0})^2\widehat{\Phi}_{h2}\,\mathrm{d}\boldsymbol{\zeta} - \frac{3}{2}(\hat{p}_{h1}\hat{u}_{ih1} + \hat{p}_{h0}\hat{u}_{ih2}) - \widehat{P}_{ijh1}\hat{u}_{jh1} - \widehat{P}_{ijh0}\hat{u}_{jh2}. \tag{16f}$$

Substituting equation (12) into equation (1), we obtain the following integral equations for the component functions $\widehat{\Phi}_{hm}$ $(m = 0, 1, \ldots)$ of the velocity distribution function $\widehat{\Phi}_H$:

$$\widehat{J}(\widehat{\Phi}_{h0}, \widehat{\Phi}_{h0}) = 0, \tag{17}$$

$$\widehat{J}(\widehat{\Phi}_{h0}, \widehat{\Phi}_{h1}) = 0, \tag{18}$$

$$2\widehat{J}(\widehat{\Phi}_{h0}, \widehat{\Phi}_{hm}) = \zeta_i \frac{\partial \widehat{\Phi}_{hm-2}}{\partial x_i} - \sum_{l=1}^{m-1} \widehat{J}(\widehat{\Phi}_{hl}, \widehat{\Phi}_{hm-l}) \quad (m \geqslant 2). \tag{19}$$

A comment is required here for the case of the BKW equation. The collision term of the BKW equation is not quadratic, and therefore its expansion is more complicated than that shown here. The left-hand side operator $2\widehat{J}(\widehat{\Phi}_{h0}, *)$ corresponds to the collision operator linearized around the Maxwellian $\widehat{\Phi}_{h0}$, but the terms on the right-hand side are more complicated and should be taken as the symbol of the term of expansion at this order as a whole. This comment applies to the whole discussions in the other sections as well as this section. For example, the left-hand side operator $2\widehat{J}(\widehat{\Phi}_{V0}, *)$ of equation (43) corresponds to the collision operator linearized around the Maxwellian $\widehat{\Phi}_{V0}$ (see, e.g., (A4), (B15)), and the $\sum$ term on the right-hand side of equation (45) is taken as the symbol. Further comments will be given for important results at each place.

The solution of equation (17) is Maxwellian:

$$\widehat{\Phi}_{h0} = \frac{\hat{\omega}_{h0}}{(\pi\hat{\tau}_{h0})^{3/2}} \exp\left(-\frac{(\zeta_i - \hat{u}_{ih0})^2}{\hat{\tau}_{h0}}\right). \tag{20}$$

The homogeneous linear integral equation (18) has five independent solutions $\widehat{\Phi}_{h0}$, $\widehat{\Phi}_{h0}\zeta_i$, and $\widehat{\Phi}_{h0}\zeta_j^2$. Thus, $\widehat{\Phi}_{h1}$ is expressed as (Grad [15]; Cercignani [14])

$$\widehat{\Phi}_{h1} = \widehat{\Phi}_{h0}\left\{\frac{\hat{\omega}_{h1}}{\hat{\omega}_{h0}} + \frac{2\hat{u}_{ih1}(\zeta_i - \hat{u}_{ih0})}{\hat{\tau}_{h0}} + \frac{\hat{\tau}_{h1}}{\hat{\tau}_{h0}}\left[\frac{(\zeta_j - \hat{u}_{jh0})^2}{\hat{\tau}_{h0}} - \frac{3}{2}\right]\right\}. \tag{21}$$

Then, the sum $\widehat{\Phi}_{h0} + \varepsilon\widehat{\Phi}_{h1}$ can be written in a Maxwellian form as follows:

$$\widehat{\Phi}_{h0} + \varepsilon\widehat{\Phi}_{h1} = \frac{\hat{\omega}_{h0} + \varepsilon\hat{\omega}_{h1}}{\pi^{3/2}(\hat{\tau}_{h0} + \varepsilon\hat{\tau}_{h1})^{3/2}} \exp\left(-\frac{[\zeta_i - (\hat{u}_{ih0} + \varepsilon\hat{u}_{ih1})]^2}{\hat{\tau}_{h0} + \varepsilon\hat{\tau}_{h1}}\right) + \mathrm{O}(\varepsilon^2). \tag{22}$$

The Hilbert solution of equation (1) is naturally Maxwellian up to the order of ε.

Equation (19) is an inhomogeneous linear integral equation for $\widehat{\Phi}_{hm}$ $(m \geqslant 2)$. The homogeneous equation corresponding to equation (19), which is the same form as equation (18), has five nontrivial solutions $\widehat{\Phi}_{h0}$, $\widehat{\Phi}_{h0}\zeta_i$, and $\widehat{\Phi}_{h0}\zeta_j^2$. Thus, the inhomogeneous term (say, Ih_m) of equation (19) should satisfy the following solvability conditions for equation (19) to have a solution:

$$\int (1, \zeta_i, \zeta_j^2) Ih_m\,\mathrm{d}\boldsymbol{\zeta} = \int (1, \zeta_i, \zeta_j^2)\zeta_k \frac{\partial \widehat{\Phi}_{hm-2}}{\partial x_k}\,\mathrm{d}\boldsymbol{\zeta} = 0 \quad (m \geqslant 2), \tag{23}$$

where the contribution of the collision integral vanishes. From the symmetry relation of the collision integral (Grad [15], Sone and Aoki [11]), each term in the summation of (19) vanishes on integration.

Equation (23) being satisfied, the solution of equation (19) is expressed in the form:

$$\widehat{\Phi}_{hm} = \widehat{\Phi}_{h0}\big(c_{m0} + c_{mi}\zeta_i + c_{m4}\zeta_j^2\big) + \widehat{\Psi}_{hm} \quad (m \geqslant 2), \tag{24}$$

where $\widehat{\Psi}_{hm}$ is the particular solution of equation (19), and c_{m0}, c_{mi}, and c_{m4} are undetermined functions of x_i and related to the macroscopic variables $\hat{\omega}_{hm}$, $\hat{u}_{ihm}$, and $\hat{\tau}_{hm}$. (These variables are determined by c_{n0}, c_{ni}, and c_{n4} $(n \leqslant m)$.) To make the expression of the relations explicit without knowing $\widehat{\Psi}_{hm}$, the following orthogonal relations may be imposed:

$$\int \big(1, \zeta_i, \zeta_j^2\big)\widehat{\Psi}_{hm}\,\mathrm{d}\boldsymbol{\zeta} = 0. \tag{25}$$

From equation (23) with $m = 2$, the following Euler set of equations is obtained:

$$\frac{\partial \hat{\omega}_{h0}\hat{u}_{ih0}}{\partial x_i} = 0, \tag{26}$$

$$\hat{\omega}_{h0}\hat{u}_{jh0}\frac{\partial \hat{u}_{ih0}}{\partial x_j} + \frac{1}{2}\frac{\partial \hat{p}_{h0}}{\partial x_i} = 0, \tag{27}$$

$$\hat{\omega}_{h0}\hat{u}_{jh0}\frac{\partial}{\partial x_j}\left(\hat{u}_{ih0}^2 + \frac{5}{2}\hat{\tau}_{h0}\right) = 0. \tag{28}$$

From equation (23) with $m = 3$, we have

$$\frac{\partial(\hat{\omega}_{h0}\hat{u}_{ih1} + \hat{\omega}_{h1}\hat{u}_{ih0})}{\partial x_i} = 0, \tag{29}$$

$$\hat{\omega}_{h0}\hat{u}_{jh0}\frac{\partial \hat{u}_{ih1}}{\partial x_j} + (\hat{\omega}_{h0}\hat{u}_{jh1} + \hat{\omega}_{h1}\hat{u}_{jh0})\frac{\partial \hat{u}_{ih0}}{\partial x_j} + \frac{1}{2}\frac{\partial \hat{p}_{h1}}{\partial x_i} = 0, \tag{30}$$

$$\hat{\omega}_{h0}\hat{u}_{jh0}\frac{\partial}{\partial x_j}\left(2\hat{u}_{ih0}\hat{u}_{ih1} + \frac{5}{2}\hat{\tau}_{h1}\right) + (\hat{\omega}_{h0}\hat{u}_{jh1} + \hat{\omega}_{h1}\hat{u}_{jh0})\frac{\partial}{\partial x_j}\left(\hat{u}_{ih0}^2 + \frac{5}{2}\hat{\tau}_{h0}\right) = 0. \tag{31}$$

We can proceed in a similar way with the information of $\widehat{\Psi}_{hm}$. Naturally the two sets of equations, (26)–(28) and (29)–(31), are derived from the Euler set of equations obtained by the original Hilbert expansion by re-expanding in the variable ε. In solving equations (26)–(28) or (29)–(31), we have to use the additional conditions (14d) or (15d). This note will not generally be repeated in the following discussions.

The stress tensor $\widehat{P}_{ijhm}$ and the heat-flow vector $\widehat{Q}_{ihm}$ are expressed by the macroscopic variables $\hat{\omega}_{hn}$, $\hat{u}_{ihn}$, and $\hat{\tau}_{hn}$ (or $\hat{p}_{hn}$) with $n \leqslant m$, their derivatives with $n \leqslant m-1$, since the velocity distribution function is expressed by them. Corresponding to the fact that the velocity distribution function is Maxwellian up to the order ε, we have

$$\widehat{P}_{ijh0} = \hat{p}_{h0}\delta_{ij}, \qquad \widehat{P}_{ijh1} = \hat{p}_{h1}\delta_{ij}, \qquad \widehat{Q}_{ih0} = 0, \qquad \widehat{Q}_{ih1} = 0. \tag{32}$$

The distribution function (up to the order ε) can be made to satisfy the diffuse reflection condition by taking $\hat{u}_{ih0} = \hat{u}_{wi}$, $\hat{\tau}_{h0} = \hat{\tau}_w$, $\hat{u}_{ih1} = 0$, and $\hat{\tau}_{h1} = 0$. However, these conditions are too strong for the two sets of equations, (26)–(28) and (29)–(31), to have solutions. (Note the condition $\hat{u}_{wi}n_i = 0$ mentioned just after equation (6) in section 2.) Thus, we look for the solution of the boundary-value problem by relaxing the restriction $\partial\widehat{\Phi}/\partial x_i = \mathrm{O}(\widehat{\Phi})$ on the Hilbert solution.

3.3. *Viscous boundary-layer solution*

In the system where the Knudsen number is small, a flow whose Mach number is of the order of unity is of a very high Reynolds number. In such a flow of a gas governed by the Navier–Stokes set of equations around a solid boundary, there appears a thin layer, called the viscous boundary layer, adjacent to the boundary. In this layer the variation of the state is strongly anisotropic, that is, the length scale of the variation in the direction parallel to the boundary is of the order of the characteristic length L, but in the direction normal to the boundary it shrinks by the factor of the inverse of the square root of the Reynolds number or the factor ε.

In view of this fact, assuming a similar situation, we will analyse the behaviour of the gas in the neighbourhood of the boundary, to obtain the correction to the Hilbert solution discussed in the previous subsection. For the convenience of the analysis near the boundary, we introduce a coordinate system (χ_1, χ_2, y):

$$x_i = \varepsilon y n_i(\chi_1, \chi_2) + x_{wi}(\chi_1, \chi_2), \tag{33}$$

where $y = 0$ is the surface of the boundary, the y coordinate (normal coordinate) is stretched by $1/\varepsilon$, and χ_1 and χ_2 coordinates are taken to be orthogonal to each other on the boundary ($y = 0$). The unit vector in the direction of the χ_α coordinate there will be denoted by $t_i^{(\alpha)}$ ($\alpha = 1$ or 2), which is a function of the two variables χ_1 and χ_2. For definiteness, the set $(t_i^{(1)}, t_i^{(2)}, n_i)$ is taken to form a right-hand system. Thus,

$$t_i^{(\alpha)} t_i^{(\beta)} = \delta_{\alpha\beta}, \qquad \left(\frac{\partial\chi_\alpha}{\partial x_i}\right)_0 \parallel t_i^{(\alpha)}, \qquad \left(\frac{\partial\chi_\alpha}{\partial x_i}\right)_0 \left(\frac{\partial\chi_\beta}{\partial x_i}\right)_0 = 0 \quad (\alpha \neq \beta), \tag{34}$$

where, and in the following analysis, the quantities in the parentheses with subscript 0 are evaluated on the boundary (or at $x_i = x_{wi}$), and the summation convention is not applied to the subscript (or superscript) α or β related to χ_α coordinate system. For simplicity, $\chi_{\alpha,\beta}$ will be used for $t_i^{(\beta)}(\partial\chi_\alpha/\partial x_i)_0$:

$$\chi_{\alpha,\beta} = t_i^{(\beta)} \left(\frac{\partial\chi_\alpha}{\partial x_i}\right)_0, \tag{35}$$

where $\chi_{\alpha,\beta} = 0$ for $\alpha \neq \beta$. In this coordinate system, the Boltzmann equation (1) is rewritten as follows:

$$\frac{1}{\varepsilon}\zeta_i n_i \frac{\partial\widehat{\Phi}}{\partial y} + \zeta_i \left(\frac{\partial\chi_1}{\partial x_i}\frac{\partial\widehat{\Phi}}{\partial\chi_1} + \frac{\partial\chi_2}{\partial x_i}\frac{\partial\widehat{\Phi}}{\partial\chi_2}\right) = \frac{1}{\varepsilon^2}\widehat{J}(\widehat{\Phi}, \widehat{\Phi}). \tag{36}$$

The solution expressing the behaviour of the gas in the layer $[\mathrm{O}(\varepsilon L)]$ is looked for in a power series of ε:

$$\widehat{\Phi}_V = \widehat{\Phi}_{V0} + \widehat{\Phi}_{V1}\varepsilon + \widehat{\Phi}_{V2}\varepsilon^2 + \cdots, \tag{37}$$

where the subscript V is attached to discriminate the type of solution. Corresponding to this expansion, we also expand the macroscopic variables $\hat{\omega}_V$, $\hat{u}_{iV}$, $\hat{\tau}_V$, $\hat{p}_V$, $\widehat{P}_{ijV}$, and $\widehat{Q}_{iV}$:

$$\hat{h}_V = \hat{h}_{V0} + \hat{h}_{V1}\varepsilon + \cdots, \tag{38}$$

where $\hat{h}_V$ represents any of the macroscopic variables. The relation of the component function $\hat{h}_{Vm}$ of $\hat{h}_V$ and $\widehat{\Phi}_{Vn}$ ($n \leqslant m$) is the same as that of $\hat{h}_{hm}$ and $\widehat{\Phi}_{hn}$ ($n \leqslant m$) (or is obtained only replacing the subscript h by V). In the following discussion, the relation is quoted by attaching the subscript V to the corresponding relation between $\hat{h}_{hm}$ and $\widehat{\Phi}_{hn}$ (equations (14a)–(16f)).

We introduce some notations for economy of space in the following analysis. The Maxwellian distribution with the macroscopic variables $\hat{\omega}_V$, $\hat{u}_{iV}$, and $\hat{\tau}_V$ is indicated by $\widehat{\Phi}_{eV}$, that is,

$$\widehat{\Phi}_{eV} = \frac{\hat{\omega}_V}{(\pi\hat{\tau}_V)^{3/2}} \exp\left(-\frac{(\zeta_i - \hat{u}_{iV})^2}{\hat{\tau}_V}\right), \tag{39}$$

and its expansion in ε, obtained by the substitution of the expansion (38), by

$$\widehat{\Phi}_{eV} = \widehat{\Phi}_{eV0} + \varepsilon\widehat{\Phi}_{eV1} + \varepsilon^2\widehat{\Phi}_{eV2} + \cdots. \tag{40}$$

Some of the explicit forms of $\widehat{\Phi}_{eVm}$ ($m = 0, 1, \ldots$) are

$$\widehat{\Phi}_{eV0} = \frac{\hat{\omega}_{V0}}{(\pi\hat{\tau}_{V0})^{3/2}} \exp\left(-\frac{(\zeta_i - \hat{u}_{iV0})^2}{\hat{\tau}_{V0}}\right) = \frac{\hat{\omega}_{V0}}{\hat{\tau}_{V0}{}^{3/2}} E(\tilde{\zeta}), \tag{41a}$$

$$\begin{aligned}\widehat{\Phi}_{eV1} &= \widehat{\Phi}_{eV0}\left[\frac{\hat{\omega}_{V1}}{\hat{\omega}_{V0}} + \frac{2(\zeta_i - \hat{u}_{iV0})}{\hat{\tau}_{V0}^{1/2}}\frac{\hat{u}_{iV1}}{\hat{\tau}_{V0}^{1/2}} + \frac{\hat{\tau}_{V1}}{\hat{\tau}_{V0}}\left(\frac{(\zeta_i - \hat{u}_{iV0})^2}{\hat{\tau}_{V0}} - \frac{3}{2}\right)\right] \\ &= \frac{\hat{\omega}_{V0}}{\hat{\tau}_{V0}{}^{3/2}} E(\tilde{\zeta})\left[\left(\frac{\hat{\omega}_{V1}}{\hat{\omega}_{V0}}\right) + 2\left(\frac{\hat{u}_{iV1}}{\hat{\tau}_{V0}^{1/2}}\right)\tilde{\zeta}_i + \left(\frac{\hat{\tau}_{V1}}{\hat{\tau}_{V0}}\right)\left(\tilde{\zeta}^2 - \frac{3}{2}\right)\right],\end{aligned} \tag{41b}$$

where

$$E(\tilde{\zeta}) = \frac{1}{\pi^{3/2}} \exp(-\tilde{\zeta}^2), \qquad \tilde{\zeta}_i = \frac{(\zeta_i - \hat{u}_{iV0})}{\hat{\tau}_{V0}^{1/2}}, \qquad \tilde{\zeta} = \sqrt{\tilde{\zeta}_i^2}.$$

It is noted that a moment of $\widehat{\Phi}_{eVm}$ with respect to ζ_i is calculated faster by taking the moment of $\widehat{\Phi}_{eV}$ and then expanding it in ε than by finding higher order $\widehat{\Phi}_{eVm}$ and then taking the moment. Thus the lengthy expression for explicit form of $\widehat{\Phi}_{eV2}$ is not shown here.

We substitute the series (37) into equation (36) and arrange the same order terms in ε. In this process, the quantities $\partial\chi_1/\partial x_i$ and $\partial\chi_2/\partial x_i$ are also expanded in the power series of y, since they are slowly varying in the variable y. Then we obtain the following series of integral equations for $\widehat{\Phi}_{Vm}$:

$$\widehat{J}(\widehat{\Phi}_{V0}, \widehat{\Phi}_{V0}) = 0, \tag{42}$$

$$2\widehat{J}(\widehat{\Phi}_{V0}, \widehat{\Phi}_{V1}) = \zeta_i n_i \frac{\partial\widehat{\Phi}_{V0}}{\partial y}, \tag{43}$$

$$2\widehat{J}(\widehat{\Phi}_{V0}, \widehat{\Phi}_{V2}) = -\widehat{J}(\widehat{\Phi}_{V1}, \widehat{\Phi}_{V1}) + \zeta_i n_i \frac{\partial\widehat{\Phi}_{V1}}{\partial y} + \zeta_i\left[\left(\frac{\partial\chi_1}{\partial x_i}\right)_0 \frac{\partial\widehat{\Phi}_{V0}}{\partial\chi_1} + \left(\frac{\partial\chi_2}{\partial x_i}\right)_0 \frac{\partial\widehat{\Phi}_{V0}}{\partial\chi_2}\right], \tag{44}$$

$$2\widehat{J}(\widehat{\Phi}_{V0}, \widehat{\Phi}_{V3}) = -\sum_{l=1}^{2}\widehat{J}(\widehat{\Phi}_{Vl}, \widehat{\Phi}_{V3-l}) + \zeta_i n_i \frac{\partial\widehat{\Phi}_{V2}}{\partial y} + \zeta_i\left[\left(\frac{\partial\chi_1}{\partial x_i}\right)_0 \frac{\partial\widehat{\Phi}_{V1}}{\partial\chi_1} + \left(\frac{\partial\chi_2}{\partial x_i}\right)_0 \frac{\partial\widehat{\Phi}_{V1}}{\partial\chi_2}\right]$$

$$+\, y\zeta_i n_j\left[\left(\frac{\partial^2\chi_1}{\partial x_i\partial x_j}\right)_0\frac{\partial\widehat{\Phi}_{V0}}{\partial\chi_1}+\left(\frac{\partial^2\chi_2}{\partial x_i\partial x_j}\right)_0\frac{\partial\widehat{\Phi}_{V0}}{\partial\chi_2}\right]. \tag{45}$$

The solution of equation (42) is the Maxwellian:

$$\widehat{\Phi}_{V0}=\widehat{\Phi}_{eV0}. \tag{46}$$

The equations for $\widehat{\Phi}_{Vm}$ $(m\geqslant 1)$ are inhomogeneous linear integral equations whose homogeneous parts are of a common form. Their common associated homogenous equation has five independent solutions: $\widehat{\Phi}_{V0}$, $\widehat{\Phi}_{V0}\zeta_i$, $\widehat{\Phi}_{V0}\zeta_j^2$. Thus, the equation for $\widehat{\Phi}_{Vm}$ $(m\geqslant 1)$ has a solution only when its inhomogeneous term Ih_m satisfies the following conditions (solvability conditions):

$$\int(1,\zeta_i,\zeta_j^2)Ih_m\,\mathrm{d}\boldsymbol{\zeta}=0\quad(m\geqslant 1). \tag{47}$$

Here, as explained in section 3.2, there is no contribution from $\widehat{J}$ terms in Ih_m. When the conditions are satisfied, the solution $\widehat{\Phi}_{Vm}$ is expressed in the following form:

$$\widehat{\Phi}_{Vm}=\widehat{\Phi}_{V0}\left(c_{m0}+c_{mi}\zeta_i+c_{m4}\zeta_j^2\right)+\widehat{\Psi}_{Vm}\quad(m\geqslant 1), \tag{48}$$

where c_{m0}, c_{mi}, and c_{m4} are undetermined functions of χ_α and y, and $\widehat{\Psi}_{Vm}$ is a particular solution of the equation.

Before proceeding to the analysis of $\widehat{\Phi}_{Vm}$ $(m\geqslant 1)$, we will make a short detour through some discussion of the boundary condition to save some manipulation. The leading term $\widehat{\Phi}_{V0}$ of the expansion (37) is made to satisfy the boundary condition (5) by assigning the boundary values of $\hat{u}_{iV0}$ and $\hat{\tau}_{V0}$ as follows:

$$\hat{u}_{iV0}=\hat{u}_{wi},\qquad \hat{\tau}_{V0}=\hat{\tau}_w,\quad \text{at } y=0. \tag{49}$$

Now we return to the discussion of $\widehat{\Phi}_{Vm}$. From the solvability conditions (47) with $m=1$, where

$$Ih_1=\zeta_i n_i\frac{\partial\widehat{\Phi}_{V0}}{\partial y}, \tag{50}$$

we obtain

$$\frac{\partial}{\partial y}(\hat{\omega}_{V0}\hat{u}_{iV0}n_i)=0, \tag{51}$$

$$\frac{\partial\hat{p}_{V0}}{\partial y}=0, \tag{52}$$

and the other three relations are reduced to identities with the aid of the following equation (53). From equations (49) and (51),

$$\hat{u}_{iV0}n_i=0, \tag{53}$$

and from equation (52), $\hat{p}_{V0}$ is a function of χ_1 and χ_2 only, i.e.

$$\hat{p}_{V0}=\hat{p}_{V0}(\chi_1,\chi_2). \tag{54}$$

When these solvability conditions are satisfied, the solution $\widehat{\Phi}_{V1}$ of equation (43) is expressed in the following form (Appendix A):

$$\widehat{\Phi}_{V1} = \widehat{\Phi}_{eV1} - \left(\frac{1}{\hat{\tau}_{V0}^{5/2}} \frac{\partial \hat{\tau}_{V0}}{\partial y}\right) \tilde{\zeta}_i n_i A(\tilde{\zeta}) E(\tilde{\zeta}) - \left(\frac{n_j}{\hat{\tau}_{V0}^2} \frac{\partial \hat{u}_{iV0}}{\partial y}\right) \left(\tilde{\zeta}_i \tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3} \delta_{ij}\right) B(\tilde{\zeta}) E(\tilde{\zeta}), \tag{55}$$

where $A(\tilde{\zeta})$ and $B(\tilde{\zeta})$ are functions of $\tilde{\zeta}$ and $\hat{\tau}_{V0}$, defined in Appendix B as solutions of integral equations related to the collision integral. There is no contribution of $\tilde{\zeta}^2 \delta_{ij}/3$ to the third term in equation (55), since $\hat{u}_{iV0} n_i = 0$. This kind of term will be eliminated without notice in the following analysis. The functions $A(\tilde{\zeta})$ and $B(\tilde{\zeta})$ are studied by Ohwada and Sone [20]. With this form of $\widehat{\Phi}_{V1}$ in the solvability conditions (47) with $m = 2$, we obtain the following set of equations:

$$\frac{\partial}{\partial y}(\hat{\omega}_{V0} \hat{u}_{iV1} n_i) = -\sum_{\alpha=1}^{2} \left(\chi_{\alpha,\alpha} \frac{\partial \hat{\omega}_{V0} \hat{u}_{iV0} t_i^{(\alpha)}}{\partial \chi_\alpha} - (-1)^\alpha g_{3-\alpha} \hat{\omega}_{V0} \hat{u}_{iV0} t_i^{(\alpha)}\right), \tag{56}$$

$$\begin{aligned} &\hat{\omega}_{V0} \left[\hat{u}_{jV1} n_j \frac{\partial \hat{u}_{iV0} t_i^{(\alpha)}}{\partial y} + \sum_{\beta=1}^{2} \hat{u}_{kV0} t_k^{(\beta)} \left(\chi_{\beta,\beta} \frac{\partial \hat{u}_{iV0} t_i^{(\alpha)}}{\partial \chi_\beta} + (-1)^\alpha g_\beta \hat{u}_{iV0} t_i^{(3-\alpha)}\right)\right] \\ &\quad = -\frac{1}{2} \chi_{\alpha,\alpha} \frac{\partial \hat{p}_{V0}}{\partial \chi_\alpha} + \frac{1}{2} \frac{\partial}{\partial y}\left(\gamma_1 \hat{\tau}_{V0}^{1/2} \frac{\partial \hat{u}_{iV0} t_i^{(\alpha)}}{\partial y}\right), \end{aligned} \tag{57}$$

$$\frac{1}{2} \frac{\partial \hat{p}_{V1}}{\partial y} = -\hat{\omega}_{V0} \sum_{\alpha=1}^{2} \left(\kappa_\alpha \left(\hat{u}_{iV0} t_i^{(\alpha)}\right)^2 + \vartheta \hat{u}_{iV0} t_i^{(\alpha)} \hat{u}_{jV0} t_j^{(3-\alpha)}\right), \tag{58}$$

$$\begin{aligned} &\frac{3}{2} \hat{\omega}_{V0} \left(\hat{u}_{jV1} n_j \frac{\partial \hat{\tau}_{V0}}{\partial y} + \sum_{\alpha=1}^{2} \hat{u}_{kV0} t_k^{(\alpha)} \chi_{\alpha,\alpha} \frac{\partial \hat{\tau}_{V0}}{\partial \chi_\alpha}\right) \\ &\quad = -\hat{p}_{V0} \left[\sum_{\alpha=1}^{2} \left(\chi_{\alpha,\alpha} \frac{\partial \hat{u}_{kV0} t_k^{(\alpha)}}{\partial \chi_\alpha} + (-1)^\alpha g_\alpha \hat{u}_{kV0} t_k^{(3-\alpha)}\right) + \frac{\partial \hat{u}_{kV1} n_k}{\partial y}\right] \\ &\qquad + \gamma_1 \hat{\tau}_{V0}^{1/2} \sum_{\alpha=1}^{2} \left(\frac{\partial \hat{u}_{kV0} t_k^{(\alpha)}}{\partial y}\right)^2 + \frac{5}{4} \frac{\partial}{\partial y}\left(\gamma_2 \hat{\tau}_{V0}^{1/2} \frac{\partial \hat{\tau}_{V0}}{\partial y}\right). \end{aligned} \tag{59}$$

For these equations the following should be noted: firstly, equations (57) and (59) are not the forms derived directly from the solvability conditions; the former is deformed with the aid of equation (56), and the latter with the aid of equations (56)–(58), to make them in a form more familiar in the classical fluid-dynamics. Secondly, κ_α, g_α, and ϑ_α are determined by the geometric properties of the boundary surface and the χ_α-coordinate curve on it:

$$\begin{aligned} &\chi_{\alpha,\alpha} \frac{\partial t_i^{(\alpha)}}{\partial \chi_\alpha} = \kappa_\alpha n_i - (-1)^\alpha g_\alpha t_i^{(3-\alpha)}, \qquad \chi_{\alpha,\alpha} \frac{\partial t_i^{(3-\alpha)}}{\partial \chi_\alpha} = (-1)^\alpha \left(-\vartheta_\alpha n_i + g_\alpha t_i^{(\alpha)}\right), \\ &\chi_{\alpha,\alpha} \frac{\partial n_i}{\partial \chi_\alpha} = -\kappa_\alpha t_i^{(\alpha)} + (-1)^\alpha \vartheta_\alpha t_i^{(3-\alpha)}, \end{aligned} \tag{60}$$

where κ_α, g_α, and ϑ_α are, respectively, the normal curvature, geodesic curvature, and geodesic torsion, multiplied by L, on the χ_α-coordinate curve (Cartan [21]). In the present case, where χ_1 and χ_2 are orthogonal,

$$\bar{\kappa} = \frac{1}{2}(\kappa_1 + \kappa_2), \quad \vartheta_1 = -\vartheta_2 = \vartheta, \tag{61}$$

where $\bar{\kappa}$ is the mean curvature, multiplied by L, of the boundary. Lastly, γ_1 and γ_2 are functions of $\hat{\tau}_{V0}$ defined in Appendix B as integrals of functions $A(\tilde{\zeta})$ and $B(\tilde{\zeta})$. They, multiplied by $\hat{\tau}_{V0}^{1/2}$, are related to viscosity and thermal conductivity of the gas at temperature $\hat{\tau}_{V0}$. For example,

$$\gamma_1 = 1.270042427, \quad \gamma_2 = 1.922284066 \text{ (hard sphere)}; \quad \gamma_1/\hat{\tau}_{V0}^{1/2} = 1, \quad \gamma_2/\hat{\tau}_{V0}^{1/2} = 1 \text{ (BKW)}. \tag{62}$$

Equations (56)–(59) being satisfied, then the solution $\widehat{\Phi}_{V2}$ of equation (44) is given in the following form (Appendix A):

$$\begin{aligned}
\widehat{\Phi}_{V2} = {} & \widehat{\Phi}_{eV2} - \left(\frac{n_i}{\hat{\tau}_{V0}^{5/2}}\frac{\partial \hat{\tau}_{V1}}{\partial y}\right)\tilde{\zeta}_i A(\tilde{\zeta})E(\tilde{\zeta}) - \left(\frac{n_j}{\hat{\tau}_{V0}^{2}}\frac{\partial \hat{u}_{iV1}}{\partial y}\right)\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)B(\tilde{\zeta})E(\tilde{\zeta}) \\
& + \left(\frac{n_j\hat{u}_{iV1}}{\hat{\tau}_{V0}^{3}}\frac{\partial \hat{\tau}_{V0}}{\partial y}\right)\left(\frac{\tilde{\zeta}_i\tilde{\zeta}_j}{\tilde{\zeta}}\frac{\partial A(\tilde{\zeta})E(\tilde{\zeta})}{\partial \tilde{\zeta}} + A(\tilde{\zeta})E(\tilde{\zeta})\delta_{ij}\right) \\
& + \left(\frac{n_i\hat{\tau}_{V1}}{\hat{\tau}_{V0}^{7/2}}\frac{\partial \hat{\tau}_{V0}}{\partial y}\right)\tilde{\zeta}_i\left(\frac{\tilde{\zeta}}{2}\frac{\partial A(\tilde{\zeta})E(\tilde{\zeta})}{\partial \tilde{\zeta}} + 3A(\tilde{\zeta})E(\tilde{\zeta}) - \hat{\tau}_{V0}\frac{\partial A(\tilde{\zeta})E(\tilde{\zeta})}{\partial \hat{\tau}_{V0}}\right) \\
& + \left(\frac{n_j\hat{u}_{kV1}}{\hat{\tau}_{V0}^{5/2}}\frac{\partial \hat{u}_{iV0}}{\partial y}\right)\left(\frac{\tilde{\zeta}_i\tilde{\zeta}_j\tilde{\zeta}_k}{\tilde{\zeta}}\frac{\partial B(\tilde{\zeta})E(\tilde{\zeta})}{\partial \tilde{\zeta}} + (\tilde{\zeta}_i\delta_{jk} + \tilde{\zeta}_j\delta_{ki})B(\tilde{\zeta})E(\tilde{\zeta})\right) \\
& + \left(\frac{n_i\hat{\tau}_{V1}}{\hat{\tau}_{V0}^{3}}\frac{\partial \hat{u}_{jV0}}{\partial y}\right)\tilde{\zeta}_i\tilde{\zeta}_j\left(\frac{\tilde{\zeta}}{2}\frac{\partial B(\tilde{\zeta})E(\tilde{\zeta})}{\partial \tilde{\zeta}} + 3B(\tilde{\zeta})E(\tilde{\zeta}) - \hat{\tau}_{V0}\frac{\partial B(\tilde{\zeta})E(\tilde{\zeta})}{\partial \hat{\tau}_{V0}}\right) \\
& - \frac{\mathcal{D}_j\hat{\tau}_{V0}}{\hat{\tau}_{V0}^{5/2}}\tilde{\zeta}_j A(\tilde{\zeta})E(\tilde{\zeta}) - \frac{\mathcal{D}_j\hat{u}_{iV0}}{\hat{\tau}_{V0}^{2}}\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)B(\tilde{\zeta})E(\tilde{\zeta}) \\
& + \frac{1}{\hat{p}_{V0}\hat{\tau}_{V0}^{3/2}}\frac{\partial^2\hat{\tau}_{V0}}{\partial y^2}\left[n_in_j\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)\mathcal{B}_1(\tilde{\zeta})E(\tilde{\zeta}) + \mathcal{N}^B(\tilde{\zeta})E(\tilde{\zeta})\right] \\
& + \frac{1}{2\hat{p}_{V0}\hat{\tau}_{V0}^{5/2}}\left(\frac{\partial\hat{\tau}_{V0}}{\partial y}\right)^2\left[n_in_j\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)\mathcal{B}_2(\tilde{\zeta})E(\tilde{\zeta}) + \mathcal{N}^A(\tilde{\zeta})E(\tilde{\zeta})\right] \\
& + \frac{n_in_j}{\hat{p}_{V0}\hat{\tau}_{V0}}\frac{\partial^2\hat{u}_{kV0}}{\partial y^2}\left[\tilde{\zeta}_i\tilde{\zeta}_j\tilde{\zeta}_k\mathcal{T}_2^A(\tilde{\zeta})E(\tilde{\zeta}) + \delta_{ij}\tilde{\zeta}_k\mathcal{T}_1^A(\tilde{\zeta})E(\tilde{\zeta})\right] \\
& + \frac{n_in_j}{\hat{p}_{V0}\hat{\tau}_{V0}^{2}}\frac{\partial\hat{\tau}_{V0}}{\partial y}\frac{\partial\hat{u}_{kV0}}{\partial y}\left[\tilde{\zeta}_i\tilde{\zeta}_j\tilde{\zeta}_k\mathcal{T}_2^B(\tilde{\zeta})E(\tilde{\zeta}) + \delta_{ij}\tilde{\zeta}_k\mathcal{T}_1^B(\tilde{\zeta})E(\tilde{\zeta})\right] \\
& + \frac{n_in_j}{\hat{p}_{V0}\hat{\tau}_{V0}^{3/2}}\frac{\partial\hat{u}_{kV0}}{\partial y}\frac{\partial\hat{u}_{lV0}}{\partial y}\left[\tilde{\zeta}_i\tilde{\zeta}_j\tilde{\zeta}_k\tilde{\zeta}_l\mathcal{Q}_4(\tilde{\zeta})E(\tilde{\zeta}) + \tilde{\zeta}_i\tilde{\zeta}_j\delta_{kl}\mathcal{Q}_3(\tilde{\zeta})E(\tilde{\zeta})\right. \\
& \qquad\qquad \left. + \delta_{ij}\tilde{\zeta}_k\tilde{\zeta}_l\mathcal{Q}_2(\tilde{\zeta})E(\tilde{\zeta}) + \delta_{ij}\delta_{kl}\mathcal{Q}_1(\tilde{\zeta})E(\tilde{\zeta})\right],
\end{aligned} \tag{63}$$

$$\mathcal{D}_i = \left(\frac{\partial\chi_1}{\partial x_i}\right)_0\frac{\partial}{\partial\chi_1} + \left(\frac{\partial\chi_2}{\partial x_i}\right)_0\frac{\partial}{\partial\chi_2},$$

where the definitions of $\mathcal{B}_1(\zeta)$, $\mathcal{T}_1^A(\zeta)$, $\mathcal{Q}_1(\zeta)$, etc. are given in Appendix B, together with $A(\zeta)$ and $B(\zeta)$. With this form of $\widehat{\Phi}_{V2}$ in the solvability conditions (47) with $m=3$, we obtain the following set of equations:

$$\begin{aligned}
&\frac{\partial}{\partial y}(\hat{\omega}_{V0}\hat{u}_{iV2}n_i+\hat{\omega}_{V1}\hat{u}_{iV1}n_i)+\sum_{\alpha=1}^{2}\chi_{\alpha,\alpha}\frac{\partial(\hat{\omega}_{V0}\hat{u}_{iV1}t_i^{(\alpha)}+\hat{\omega}_{V1}\hat{u}_{iV0}t_i^{(\alpha)})}{\partial\chi_\alpha}\\
&\quad-2\bar{\kappa}\hat{\omega}_{V0}\hat{u}_{iV1}n_i-\sum_{\alpha=1}^{2}(-1)^\alpha g_{3-\alpha}\big(\hat{\omega}_{V0}\hat{u}_{iV1}t_i^{(\alpha)}+\hat{\omega}_{V1}\hat{u}_{iV0}t_i^{(\alpha)}\big)\\
&\quad+\sum_{\alpha=1}^{2}y\left(\kappa_\alpha\chi_{\alpha,\alpha}\frac{\partial\hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(\alpha)}}{\partial\chi_\alpha}+\vartheta\chi_{\alpha,\alpha}\frac{\partial\hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(3-\alpha)}}{\partial\chi_\alpha}\right)\\
&\quad-\sum_{\alpha=1}^{2}(-1)^\alpha y(\kappa_{3-\alpha}g_{3-\alpha}+\vartheta g_\alpha)\hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(\alpha)}=0,
\end{aligned}\tag{64}$$

$$\begin{aligned}
&\hat{\omega}_{V0}\hat{u}_{jV1}n_j\frac{\partial\hat{u}_{iV1}t_i^{(\alpha)}}{\partial y}+(\hat{\omega}_{V0}\hat{u}_{jV2}n_j+\hat{\omega}_{V1}\hat{u}_{jV1}n_j)\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}\\
&\quad+\sum_{\beta=1}^{2}\hat{\omega}_{V0}\hat{u}_{jV0}t_j^{(\beta)}\chi_{\beta,\beta}\frac{\partial\hat{u}_{iV1}t_i^{(\alpha)}}{\partial\chi_\beta}-\hat{\omega}_{V0}\hat{u}_{iV1}n_i\big(\kappa_\alpha\hat{u}_{jV0}t_j^{(\alpha)}+\vartheta\hat{u}_{jV0}t_j^{(3-\alpha)}\big)\\
&\quad+(-1)^\alpha\hat{\omega}_{V0}\hat{u}_{jV1}t_j^{(3-\alpha)}\left(\sum_{\beta=1}^{2}g_\beta\hat{u}_{iV0}t_i^{(\beta)}\right)\\
&\quad+\sum_{\beta=1}^{2}\big(\hat{\omega}_{V0}\hat{u}_{jV1}t_j^{(\beta)}+\hat{\omega}_{V1}\hat{u}_{jV0}t_j^{(\beta)}\big)\chi_{\beta,\beta}\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial\chi_\beta}\\
&\quad+(-1)^\alpha\hat{u}_{iV0}t_i^{(3-\alpha)}\left(\sum_{\beta=1}^{2}g_\beta\big(\hat{\omega}_{V0}\hat{u}_{jV1}t_j^{(\beta)}+\hat{\omega}_{V1}\hat{u}_{jV0}t_j^{(\beta)}\big)\right)\\
&\quad+\sum_{\beta=1}^{2}y\hat{\omega}_{V0}\big(\kappa_\beta\hat{u}_{jV0}t_j^{(\beta)}+\vartheta\hat{u}_{jV0}t_j^{(3-\beta)}\big)\chi_{\beta,\beta}\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial\chi_\beta}\\
&\quad+(-1)^\alpha y\hat{\omega}_{V0}\left(\sum_{\beta=1}^{2}\big(\kappa_\beta g_\beta+\vartheta g_{3-\beta}\big)\hat{u}_{iV0}t_i^{(\beta)}\right)\hat{u}_{jV0}t_j^{(3-\alpha)}\\
&=-\frac{1}{2}\chi_{\alpha,\alpha}\frac{\partial\hat{p}_{V1}}{\partial\chi_\alpha}-\frac{1}{2}y\left(\kappa_\alpha\chi_{\alpha,\alpha}\frac{\partial\hat{p}_{V0}}{\partial\chi_\alpha}+\vartheta\chi_{3-\alpha,3-\alpha}\frac{\partial\hat{p}_{V0}}{\partial\chi_{3-\alpha}}\right)\\
&\quad+\frac{1}{2}\frac{\partial}{\partial y}\left[\gamma_1\hat{\tau}_{V0}^{1/2}\left(\frac{\partial\hat{u}_{iV1}t_i^{(\alpha)}}{\partial y}+\kappa_\alpha\hat{u}_{iV0}t_i^{(\alpha)}+\vartheta\hat{u}_{iV0}t_i^{(3-\alpha)}\right)+\hat{\tau}_{V1}\frac{\mathrm{d}\gamma_1\hat{\tau}_{V0}^{1/2}}{\mathrm{d}\hat{\tau}_{V0}}\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}\right]\\
&\quad-\frac{1}{2}\gamma_1\hat{\tau}_{V0}^{1/2}\left[(2\bar{\kappa}+\kappa_\alpha)\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}+\vartheta\frac{\partial\hat{u}_{iV0}t_i^{(3-\alpha)}}{\partial y}\right],
\end{aligned}\tag{65}$$

$$\sum_{\alpha=1}^{2}\hat{\omega}_{V0}\hat{u}_{jV0}t_j^{(\alpha)}\left(\chi_{\alpha,\alpha}\frac{\partial\hat{u}_{iV1}n_i}{\partial\chi_\alpha}+\kappa_\alpha\hat{u}_{iV1}t_i^{(\alpha)}+\vartheta\hat{u}_{iV1}t_i^{(3-\alpha)}\right)+\hat{\omega}_{V0}\hat{u}_{jV1}n_j\frac{\partial\hat{u}_{iV1}n_i}{\partial y}$$

$$
\begin{aligned}
&+\sum_{\alpha=1}^{2}\big(\hat{\omega}_{V0}\hat{u}_{jV1}t_j^{(\alpha)}+\hat{\omega}_{V1}\hat{u}_{jV0}t_j^{(\alpha)}\big)\big(\kappa_\alpha\hat{u}_{iV0}t_i^{(\alpha)}+\vartheta\hat{u}_{iV0}t_i^{(3-\alpha)}\big)\\
&+y\hat{\omega}_{V0}\sum_{\alpha=1}^{2}\big(\kappa_\alpha\hat{u}_{iV0}t_i^{(\alpha)}+\vartheta\hat{u}_{iV0}t_i^{(3-\alpha)}\big)^2\\
=&-\frac{1}{2}\frac{\partial\hat{p}_{V2}}{\partial y}+\frac{1}{2}\sum_{\alpha=1}^{2}\left[\chi_{\alpha,\alpha}\frac{\partial}{\partial\chi_\alpha}\left(\gamma_1\hat{\tau}_{V0}^{1/2}\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}\right)+(-1)^\alpha g_\alpha\gamma_1\hat{\tau}_{V0}^{1/2}\frac{\partial\hat{u}_{iV0}t_i^{(3-\alpha)}}{\partial y}\right]\\
&+\frac{2}{3}\frac{\partial}{\partial y}\left\{\gamma_1\hat{\tau}_{V0}^{1/2}\left[\frac{\partial\hat{u}_{iV1}n_i}{\partial y}-\frac{1}{2}\sum_{\alpha=1}^{2}\left(\chi_{\alpha,\alpha}\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial\chi_\alpha}+(-1)^\alpha g_\alpha\hat{u}_{iV0}t_i^{(3-\alpha)}\right)\right]\right\}\\
&-\frac{1}{3\hat{p}_{V0}}\frac{\partial}{\partial y}\left[\gamma_3\hat{\tau}_{V0}\frac{\partial^2\hat{\tau}_{V0}}{\partial y^2}+\gamma_7\left(\frac{\partial\hat{\tau}_{V0}}{\partial y}\right)^2\right]-\frac{\partial}{\partial y}\left[\frac{\gamma_8-2\gamma_9}{3\hat{\omega}_{V0}}\sum_{\alpha=1}^{2}\left(\frac{\partial\hat{u}_{jV0}t_j^{(\alpha)}}{\partial y}\right)^2\right],
\end{aligned}\tag{66}
$$

$$
\begin{aligned}
&\frac{3}{2}\left[\hat{\omega}_{V0}\hat{u}_{jV1}n_j\frac{\partial\hat{\tau}_{V1}}{\partial y}+(\hat{\omega}_{V0}\hat{u}_{jV2}n_j+\hat{\omega}_{V1}\hat{u}_{jV1}n_j)\frac{\partial\hat{\tau}_{V0}}{\partial y}+\hat{\omega}_{V0}\hat{u}_{jV0}\sum_{\alpha=1}^{2}\chi_{\alpha,\alpha}t_j^{(\alpha)}\frac{\partial\hat{\tau}_{V1}}{\partial\chi_\alpha}\right]\\
&\quad+\frac{3}{2}\sum_{\alpha=1}^{2}\chi_{\alpha,\alpha}\big[\hat{\omega}_{V0}\hat{u}_{jV1}t_j^{(\alpha)}+\hat{\omega}_{V1}\hat{u}_{jV0}t_j^{(\alpha)}+y\hat{\omega}_{V0}\big(\kappa_\alpha\hat{u}_{jV0}t_j^{(\alpha)}+\vartheta\hat{u}_{jV0}t_j^{(3-\alpha)}\big)\big]\frac{\partial\hat{\tau}_{V0}}{\partial\chi_\alpha}\\
=&-\hat{p}_{V0}\left[\frac{\partial\hat{u}_{jV2}n_j}{\partial y}+\sum_{\alpha=1}^{2}\left(\chi_{\alpha,\alpha}\frac{\partial\hat{u}_{jV1}t_j^{(\alpha)}}{\partial\chi_\alpha}-(-1)^\alpha g_{3-\alpha}\hat{u}_{jV1}t_j^{(\alpha)}\right)-2\bar{\kappa}\hat{u}_{jV1}n_j\right.\\
&\quad\left.+\sum_{\alpha=1}^{2}y\left(\kappa_\alpha\chi_{\alpha,\alpha}\frac{\partial\hat{u}_{jV0}t_j^{(\alpha)}}{\partial\chi_\alpha}+\vartheta\chi_{\alpha,\alpha}\frac{\partial\hat{u}_{jV0}t_j^{(3-\alpha)}}{\partial\chi_\alpha}-(-1)^\alpha(\kappa_{3-\alpha}g_{3-\alpha}+\vartheta g_\alpha)\hat{u}_{jV0}t_j^{(\alpha)}\right)\right]\\
&\quad-\hat{p}_{V1}\left[\frac{\partial\hat{u}_{jV1}n_j}{\partial y}+\sum_{\alpha=1}^{2}\left(\chi_{\alpha,\alpha}\frac{\partial\hat{u}_{jV0}t_j^{(\alpha)}}{\partial\chi_\alpha}-(-1)^\alpha g_{3-\alpha}\hat{u}_{jV0}t_j^{(\alpha)}\right)\right]\\
&\quad+\sum_{\alpha=1}^{2}\left[2\gamma_1\hat{\tau}_{V0}^{1/2}\left(\frac{\partial\hat{u}_{jV1}t_j^{(\alpha)}}{\partial y}+\kappa_\alpha\hat{u}_{jV0}t_j^{(\alpha)}+\vartheta\hat{u}_{jV0}t_j^{(3-\alpha)}\right)\right.\\
&\quad\left.+\hat{\tau}_{V1}\frac{\mathrm{d}\gamma_1\hat{\tau}_{V0}^{1/2}}{\mathrm{d}\hat{\tau}_{V0}}\frac{\partial\hat{u}_{jV0}t_j^{(\alpha)}}{\partial y}\right]\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}\\
&\quad+\frac{5}{4}\frac{\partial}{\partial y}\left(\gamma_2\hat{\tau}_{V0}^{1/2}\frac{\partial\hat{\tau}_{V1}}{\partial y}+\hat{\tau}_{V1}\frac{\mathrm{d}\gamma_2\hat{\tau}_{V0}^{1/2}}{\mathrm{d}\hat{\tau}_{V0}}\frac{\partial\hat{\tau}_{V0}}{\partial y}\right)-\frac{5}{2}\bar{\kappa}\gamma_2\hat{\tau}_{V0}^{1/2}\frac{\partial\hat{\tau}_{V0}}{\partial y},
\end{aligned}\tag{67}
$$

where γ_3, γ_7, γ_8, and γ_9, as well as γ_1 and γ_2, are functions of $\hat{\tau}_{V0}$ defined in Appendix B.

The series of equations obtained from the solvability conditions (47) constitutes systems of equations that determine the component functions $\hat{\omega}_{Vm}$, $\hat{u}_{iVm}$, $\hat{\tau}_{Vm}$, and $\hat{p}_{Vm}$ of the macroscopic variables $\hat{\omega}_V$, $\hat{u}_{iV}$, $\hat{\tau}_V$, and $\hat{p}_V$. That is, equations (51), (52), (57), and (59) with equation (56) determine $\hat{\omega}_{V0}$, $\hat{u}_{iV0}$, $\hat{\tau}_{V0}$, and $\hat{p}_{V0}$; equations (56), (58), (65), and (67) with equation (64) determine $\hat{\omega}_{V1}$, $\hat{u}_{iV1}$, $\hat{\tau}_{V1}$, and $\hat{p}_{V1}$. We can obtain the higher-order equations in a similar way. Corresponding to the degeneration of three relations in the solvability conditions (47) with $m=1$ to identities, the arrangement of equations from the solvability conditions is staggered, but it is consistent. In the leading system $(m=0)$, we already have the boundary condition (49), so called non-slip condition, on the boundary $y=0$, with the aid of which equation (53) is obtained. In the next subsection, we will discuss the boundary condition of the next system ($m=1$).

The expressions of $\widehat{P}_{ijVm}$ and $\widehat{Q}_{iVm}$by $\hat{\omega}_{Vn}$, $\hat{u}_{iVn}$, and $\hat{\tau}_{Vn}$ (or $\hat{p}_{Vn}$) ($n \leqslant m$) are

$$\widehat{P}_{ijV0} = \hat{p}_{V0}\delta_{ij},$$

$$\widehat{P}_{ijV1}n_i n_j = \widehat{P}_{ijV1}t_i^{(\alpha)}t_j^{(\alpha)} = \hat{p}_{V1}, \qquad \widehat{P}_{ijV1}n_i t_j^{(\alpha)} = -\gamma_1\hat{\tau}_{V0}^{1/2}\frac{\partial \hat{u}_{jV0}t_j^{(\alpha)}}{\partial y}, \qquad \widehat{P}_{ijV1}t_i^{(\alpha)}t_j^{(3-\alpha)} = 0,$$

$$\begin{aligned}
\widehat{P}_{ijV2}n_i n_j &= \hat{p}_{V2} - \frac{4}{3}\gamma_1\hat{\tau}_{V0}^{1/2}\left[\frac{\partial \hat{u}_{jV1}n_j}{\partial y} - \frac{1}{2}\sum_{\alpha=1}^{2}\left(\chi_{\alpha,\alpha}\frac{\partial \hat{u}_{jV0}t_j^{(\alpha)}}{\partial \chi_\alpha} + (-1)^\alpha g_\alpha \hat{u}_{jV0}t_j^{(3-\alpha)}\right)\right] \\
&\quad + \frac{2}{3\hat{p}_{V0}}\left[\gamma_3\hat{\tau}_{V0}\frac{\partial^2\hat{\tau}_{V0}}{\partial y^2} + \gamma_7\left(\frac{\partial\hat{\tau}_{V0}}{\partial y}\right)^2\right] + \frac{2(\gamma_8 - 2\gamma_9)}{3\hat{\omega}_{V0}}\sum_{\alpha=1}^{2}\left(\frac{\partial \hat{u}_{jV0}t_j^{(\alpha)}}{\partial y}\right)^2,
\end{aligned}$$

$$\begin{aligned}
\widehat{P}_{ijV2}t_i^{(\alpha)}t_j^{(\alpha)} &= \hat{p}_{V2} + \frac{2}{3}\gamma_1\hat{\tau}_{V0}^{1/2}\frac{\partial \hat{u}_{jV1}n_j}{\partial y} - 2\gamma_1\hat{\tau}_{V0}^{1/2}\left(\chi_{\alpha,\alpha}\frac{\partial \hat{u}_{jV0}t_j^{(\alpha)}}{\partial \chi_\alpha} + (-1)^\alpha g_\alpha \hat{u}_{jV0}t_j^{(3-\alpha)}\right) \\
&\quad + \frac{2}{3}\gamma_1\hat{\tau}_{V0}^{1/2}\sum_{\beta=1}^{2}\left(\chi_{\beta,\beta}\frac{\partial \hat{u}_{jV0}t_j^{(\beta)}}{\partial \chi_\beta} + (-1)^\beta g_\beta \hat{u}_{jV0}t_j^{(3-\beta)}\right) \\
&\quad - \frac{1}{3\hat{p}_{V0}}\left[\gamma_3\hat{\tau}_{V0}\frac{\partial^2\hat{\tau}_{V0}}{\partial y^2} + \gamma_7\left(\frac{\partial\hat{\tau}_{V0}}{\partial y}\right)^2\right] + \frac{2\gamma_8}{\hat{\omega}_{V0}}\left(\frac{\partial \hat{u}_{jV0}t_j^{(\alpha)}}{\partial y}\right)^2 \\
&\quad + \frac{2(\gamma_9 - 2\gamma_8)}{3\hat{\omega}_{V0}}\sum_{\beta=1}^{2}\left(\frac{\partial \hat{u}_{jV0}t_j^{(\beta)}}{\partial y}\right)^2,
\end{aligned}$$

$$\widehat{P}_{ijV2}n_i t_j^{(\alpha)} = -\gamma_1\hat{\tau}_{V0}^{1/2}\left(\frac{\partial \hat{u}_{jV1}t_j^{(\alpha)}}{\partial y} + \kappa_\alpha \hat{u}_{jV0}t_j^{(\alpha)} + \vartheta \hat{u}_{jV0}t_j^{(3-\alpha)}\right) - \hat{\tau}_{V1}\frac{d\gamma_1\hat{\tau}_{V0}^{1/2}}{d\hat{\tau}_{V0}}\frac{\partial \hat{u}_{jV0}t_j^{(\alpha)}}{\partial y},$$

$$\widehat{P}_{ijV2}t_i^{(\alpha)}t_j^{(3-\alpha)} = -\gamma_1\hat{\tau}_{V0}^{1/2}\sum_{\beta=1}^{2}\left(\chi_{\beta,\beta}\frac{\partial \hat{u}_{jV0}t_j^{(3-\beta)}}{\partial \chi_\beta} - (-1)^\beta g_\beta \hat{u}_{jV0}t_j^{(\beta)}\right) + \frac{2\gamma_8}{\hat{\omega}_{V0}}\frac{\partial \hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}\frac{\partial \hat{u}_{jV0}t_j^{(3-\alpha)}}{\partial y}, \tag{68}$$

$$\widehat{Q}_{iV0} = 0, \qquad \widehat{Q}_{iV1}n_i = -\frac{5}{4}\gamma_2\hat{\tau}_{V0}^{1/2}\frac{\partial\hat{\tau}_{V0}}{\partial y}, \qquad \widehat{Q}_{iV1}t_i^{(\alpha)} = 0,$$

$$\widehat{Q}_{iV2}n_i = -\frac{5}{4}\left(\gamma_2\hat{\tau}_{V0}^{1/2}\frac{\partial\hat{\tau}_{V1}}{\partial y} + \hat{\tau}_{V1}\frac{d\gamma_2\hat{\tau}_{V0}^{1/2}}{d\hat{\tau}_{V0}}\frac{\partial\hat{\tau}_{V0}}{\partial y}\right),$$

$$\widehat{Q}_{iV2}t_i^{(\alpha)} = -\frac{5}{4}\gamma_2\hat{\tau}_{V0}^{1/2}\chi_{\alpha,\alpha}\frac{\partial\hat{\tau}_{V0}}{\partial \chi_\alpha} + \frac{\gamma_3\hat{\tau}_{V0}}{2\hat{\omega}_{V0}}\frac{\partial^2\hat{u}_{iV0}t_i^{(\alpha)}}{\partial y^2} + \frac{4\gamma_{10}}{\hat{\omega}_{V0}}\frac{\partial\hat{\tau}_{V0}}{\partial y}\frac{\partial \hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}, \tag{69}$$

where γ_{10}, as well as γ_1, γ_2, etc., is a function of $\hat{\tau}_{V0}$ and is defined in Appendix B.

The terms with the factor γ_1 in $\widehat{P}_{ijVm}$ are the viscous stress given by the Newton law, and the terms with factor γ_2 in $\widehat{Q}_{iVm}$ are the heat flow by the Fourier law. The $\gamma_1\hat{\tau}_{V0}^{1/2}$ and $\gamma_2\hat{\tau}_{V0}^{1/2}$ are, respectively, the (nondimensional) viscosity and thermal conductivity of the gas, and $\hat{\tau}_{V1}d\gamma_1\hat{\tau}_{V0}^{1/2}/d\hat{\tau}_{V0}$ in $\widehat{P}_{ijV2}$ and $\hat{\tau}_{V1}d\gamma_2\hat{\tau}_{V0}^{1/2}/d\hat{\tau}_{V0}$ in $\widehat{Q}_{iV2}$ are due to their temperature dependence. The complicated forms (68) and (69) are due to the curved coordinates and the dependence of the viscosity and thermal conductivity on the temperature of the gas. The other terms are due to non-Navier–Stokes stress and heat flow. Among them, the terms with factor γ_3 and γ_7 in the stress formulas are called thermal stress. The stress tensor and heat-flow vector up to the second order, i.e., $\widehat{P}_{ijV2}$ and $\widehat{Q}_{iV2}$, contribute to equations (65)–(67), that is, the information of $\widehat{\Phi}_{V2}$ is required to derive equations (65)–(67). However, only the γ_1 and γ_2 terms contribute to equations (65) and (67). As we have discussed, these equations, (65) and (67), determine $\hat{\omega}_{V1}$, $\hat{u}_{iV1}$, $\hat{\tau}_{V1}$, and $\hat{p}_{V1}$ with the aid of equation (64). That is, these variables (thus, $\widehat{\Phi}_{V1}$ with the aid of equation (55)) are determined by the equations without non-Navier–Stokes stress and heat flow, which disagrees with the statement of Darrozes [6]. Incidentally, equation (66), one of the equations that determine the next-order macroscopic variables, contains non-Navier–Stokes terms.

3.4. Knudsen-layer solution and slip condition

The boundary condition (5) is matched with the first term $\widehat{\Phi}_{V0}$ by imposing the nonslip condition (49). Therefore, the higher order term $\widehat{\Phi}_{Vm}$ ($m \geqslant 1$) should satisfy the same form of condition as equation (5), with $\widehat{\Phi}$ in $\hat{\sigma}_w$ replaced by $\widehat{\Phi}_{Vm}$, for the solution $\widehat{\Phi}_V$ to satisfy the boundary condition. However, the solution $\widehat{\Phi}_V$ does not have this freedom. In fact, the functional form of $\widehat{\Phi}_{V1}$, as a function of ζ_i, is different from that of the boundary condition owing to the second and third terms of equation (55). These two terms are already determined by the macroscopic variables $\hat{u}_{iV0}$, $\hat{\tau}_{V0}$, $\partial\hat{\tau}_{V0}/\partial y$, and $\partial\hat{u}_{iV0}/\partial y$ obtained from equations (14d)$_V$, (54), (56), (57), and (59) under the boundary condition (49) at $y = 0$ and that at $y = \infty$ to be discussed later, and they generally are not zero. Thus, we cannot construct the solution satisfying the boundary condition at $y = 0$ with $\widehat{\Phi}_V$. This can also be seen by the fact that equation (36) for $\widehat{\Phi}_V$ is of a singular type, where the differential term $\partial/\partial y$ is, in practice, multiplied by the small parameter ε. In view of this character of equation (36), we assume that the state of the gas varies appreciably in the direction normal to the boundary over the distance of $y = \mathrm{O}(\varepsilon)$ in the neighbourhood of the boundary, and try to find the solution of the boundary-value problem by modifying $\widehat{\Phi}_V$ only in the neighbourhood of the boundary. That is, we put the solution in the form:

$$\widehat{\Phi} = \widehat{\Phi}_V + \widehat{\Phi}_K, \tag{70}$$

where the correction term $\widehat{\Phi}_K$ is assumed to vary appreciably in the direction n_i over the distance of the order of the mean free path l_0 or $y = \mathrm{O}(\varepsilon)$ and to be appreciable only in the neighbourhood (or in $y = \mathrm{O}(\varepsilon)$) of the boundary, that is, $\partial\widehat{\Phi}_K/\partial y = \mathrm{O}(\widehat{\Phi}_K/\varepsilon)$, and $\widehat{\Phi}_K$ vanishes very rapidly (or faster than any inverse power of y/ε) as y/ε tends to infinity (Knudsen-layer correction). For the convenience of analysis, we further stretch the coordinate y by $1/\varepsilon$:

$$x_i = \varepsilon^2 \eta n_i(\chi_1, \chi_2) + x_{wi}(\chi_1, \chi_2). \tag{71}$$

Then, the solution is conveniently expressed by the two sets (χ_1, χ_2, y) and (χ_1, χ_2, η) of variables:

$$\widehat{\Phi} = \widehat{\Phi}_V(\chi_1, \chi_2, y) + \widehat{\Phi}_K(\chi_1, \chi_2, \eta). \tag{72}$$

The Knudsen-layer correction $\widehat{\Phi}_K(\chi_1, \chi_2, \eta)$ is also expanded in power series of ε:

$$\widehat{\Phi}_K = \widehat{\Phi}_{K1}\varepsilon + \widehat{\Phi}_{K2}\varepsilon^2 + \cdots, \tag{73}$$

where the series starts from the term of order ε, since $\widehat{\Phi}_{V0}$ satisfies the boundary condition at $y = 0$ by imposing equation (49).

In the new coordinate system (χ_1, χ_2, η), the Boltzmann equation (1) or (36) is rewritten as follows:

$$\zeta_i n_i \frac{\partial\widehat{\Phi}}{\partial\eta} + \varepsilon^2 \zeta_i \left(\frac{\partial\chi_1}{\partial x_i}\frac{\partial\widehat{\Phi}}{\partial\chi_1} + \frac{\partial\chi_2}{\partial x_i}\frac{\partial\widehat{\Phi}}{\partial\chi_2}\right) = \widehat{J}(\widehat{\Phi}, \widehat{\Phi}). \tag{74}$$

Equation (72) with the expansions (37) and (73) is substituted into the Boltzmann equation (74), and the same order terms of ε are arranged. In this process, the fact that $\widehat{\Phi}_V$ satisfies the Boltzmann equation is used, and the series expansion of $\widehat{\Phi}_V$ in power series of y:

$$\widehat{\Phi}_V = \big(\widehat{\Phi}_{V0}\big)_0 + \varepsilon\big[\big(\widehat{\Phi}_{V1}\big)_0 + \eta\big(\partial\widehat{\Phi}_{V0}/\partial y\big)_0\big] + \cdots, \tag{75}$$

is applied to the product of $\widehat{\Phi}_V$ and $\widehat{\Phi}_K$ in the collision term, since $\widehat{\Phi}_K$ is rapidly decaying. In equation (75), the quantities in the parentheses with subscript 0 are evaluated at $y = 0$. As the result, a series of equations for $\widehat{\Phi}_{Km}$ ($m \geqslant 1$) is obtained as follows:

$$\zeta_i n_i \frac{\partial\widehat{\Phi}_{K1}}{\partial\eta} = 2\widehat{J}\big(\big(\widehat{\Phi}_{V0}\big)_0, \widehat{\Phi}_{K1}\big), \tag{76}$$

$$\zeta_i n_i \frac{\partial\widehat{\Phi}_{K2}}{\partial\eta} = 2\widehat{J}\big(\big(\widehat{\Phi}_{V0}\big)_0, \widehat{\Phi}_{K2}\big) + 2\widehat{J}\big(\big(\widehat{\Phi}_{V1}\big)_0 + \eta\big(\partial\widehat{\Phi}_{V0}/\partial y\big)_0, \widehat{\Phi}_{K1}\big) + \widehat{J}\big(\widehat{\Phi}_{K1}, \widehat{\Phi}_{K1}\big). \tag{77}$$

Equation (76) is the one-dimensional Boltzmann equation linearized around the Maxwellian $(\widehat{\Phi}_{V0})_0$, where $\hat{u}_{iV0} = \hat{u}_{wi}$ and $\hat{\tau}_{V0} = \hat{\tau}_w$.

The boundary condition for $\widehat{\Phi}_{Km}$ is given as follows. The Knudsen-layer correction $\widehat{\Phi}_K$ is introduced as the correction to $\widehat{\Phi}_V$ in the neighbourhood of the boundary of $\eta = \mathrm{O}(1)$ and is assumed to vanish faster than any inverse power of η. Thus,

$$\widehat{\Phi}_{Km} \to 0, \quad \text{as } \eta \to \infty. \tag{78}$$

(The speed of decay is verified for a hard-sphere gas by Bardos et al. [22].) From equations (5) and (70), $\widehat{\Phi}$ (or $\widehat{\Phi}_V + \widehat{\Phi}_K$) should satisfy the following condition:

$$\widehat{\Phi}_V + \widehat{\Phi}_K = \frac{\hat{\sigma}_w}{(\pi\hat{\tau}_w)^{3/2}} \exp\left(-\frac{(\zeta_i - \hat{u}_{wi})^2}{\hat{\tau}_w}\right) \quad (\zeta_i n_i > 0), \text{ at } \eta = 0,$$

where $\hat{\sigma}_w$ is given by equation (6) with $\widehat{\Phi} = \widehat{\Phi}_V + \widehat{\Phi}_K$. On the other hand, by the choice (49), at $y = 0$,

$$\widehat{\Phi}_{V0} = (\widehat{\Phi}_{V0})_0 = \frac{(\hat{\omega}_{V0})_0}{(\pi\hat{\tau}_w)^{3/2}} \exp\left(-\frac{(\zeta_i - \hat{u}_{wi})^2}{\hat{\tau}_w}\right). \tag{79}$$

Therefore, at $\eta = 0$,

$$\widehat{\Phi}_{Vm} + \widehat{\Phi}_{Km} = \frac{\hat{\sigma}_{wm}}{(\pi\hat{\tau}_w)^{3/2}} \exp\left(-\frac{(\zeta_i - \hat{u}_{wi})^2}{\hat{\tau}_w}\right) \quad (\zeta_i n_i > 0), \text{ for } m \geqslant 1, \tag{80}$$

where

$$\hat{\sigma}_{wm} = -2\sqrt{\frac{\pi}{\hat{\tau}_w}} \int_{\zeta_i n_i < 0} \zeta_j n_j \big(\widehat{\Phi}_{Vm} + \widehat{\Phi}_{Km}\big)\, \mathrm{d}\boldsymbol{\zeta}. \tag{81}$$

Specifically, the boundary condition for $\widehat{\Phi}_{K1}$ at $\eta = 0$ is given as follows:

$$\widehat{\Phi}_{K1} = -\widehat{\Phi}_{V0}\bigg[\frac{\hat{\omega}_{V1} - \hat{\sigma}_{w1}}{\hat{\omega}_{V0}} + \frac{2(\zeta_i - \hat{u}_{wi})\hat{u}_{iV1}}{\hat{\tau}_w} + \frac{\hat{\tau}_{V1}}{\hat{\tau}_w}\left(\frac{(\zeta_i - \hat{u}_{wi})^2}{\hat{\tau}_w} - \frac{3}{2}\right)$$
$$- \frac{\zeta_i n_i A\big(\sqrt{(\zeta_j - \hat{u}_{wj})^2/\hat{\tau}_w}\big)}{\hat{\omega}_{V0}\hat{\tau}_w^{3/2}} \frac{\partial\hat{\tau}_{V0}}{\partial y}$$

$$- \frac{(\zeta_i - \hat{u}_{wi})\zeta_j n_j B(\sqrt{(\zeta_k - \hat{u}_{wk})^2/\hat{\tau}_w})}{\hat{\omega}_{V0}\hat{\tau}_w^{3/2}} \frac{\partial \hat{u}_{iV0}}{\partial y}\Bigg] \quad (\zeta_i n_i > 0), \tag{82}$$

$$\hat{\sigma}_{w1} = \hat{\omega}_{V1} + \frac{\hat{\omega}_{V0}\hat{\tau}_{V1}}{2\hat{\tau}_w} - \sqrt{\frac{\pi}{\hat{\tau}_w}}\left(\hat{\omega}_{V0}\hat{u}_{iV1}n_i + 2\int_{\zeta_i n_i < 0} \zeta_j n_j \widehat{\Phi}_{K1}\,\mathrm{d}\boldsymbol{\zeta}\right), \tag{83}$$

where the relations (49), (53), and $\hat{u}_{wi}n_i = 0$ and the subsidiary condition in equation (B1) are used.

The Knudsen-layer part ($\hat{\omega}_K$, $\hat{u}_{iK}$, $\hat{\tau}_K$, $\hat{p}_K$, $\widehat{P}_{ijK}$, $\widehat{Q}_{iK}$) of the macroscopic variables ($\hat{\omega}$, $\hat{u}_i$, $\hat{\tau}$, $\hat{p}$, $\widehat{P}_{ij}$, $\widehat{Q}_i$) are defined as the remainders ($\hat{\omega} - \hat{\omega}_V$, $\hat{u}_i - \hat{u}_{iV}$, $\hat{\tau} - \hat{\tau}_V$, etc.). Then, they depend on $\widehat{\Phi}_V$ as well as $\widehat{\Phi}_K$, since the relations between the macroscopic variables and the velocity distribution functions, i.e. equations (7a)–(7f), are nonlinear. For example,

$$(\hat{\omega}_V + \hat{\omega}_K)\hat{u}_{iK} = \int \zeta_i \widehat{\Phi}_K\,\mathrm{d}\boldsymbol{\zeta} - \hat{u}_{iV}\hat{\omega}_K. \tag{84}$$

Corresponding to the expansion (73), the Knudsen-layer part of the macroscopic variables is also expanded in a power series of ε. The relations of their component functions of the expansion to $\widehat{\Phi}_{K1}$ are given as follows:

$$\begin{aligned}
&\hat{\omega}_{K1} = \int \widehat{\Phi}_{K1}\,\mathrm{d}\boldsymbol{\zeta}, \qquad (\hat{\omega}_{V0})_0\hat{u}_{iK1} = \int \zeta_i \widehat{\Phi}_{K1}\,\mathrm{d}\boldsymbol{\zeta} - \hat{\omega}_{K1}(\hat{u}_{iV0})_0,\\
&\frac{3}{2}(\hat{\omega}_{V0})_0\hat{\tau}_{K1} = \int \big[\zeta_j - (\hat{u}_{jV0})_0\big]^2 \widehat{\Phi}_{K1}\,\mathrm{d}\boldsymbol{\zeta} - \frac{3}{2}\hat{\omega}_{K1}(\hat{\tau}_{V0})_0,\\
&\hat{p}_{K1} = (\hat{\omega}_{V0})_0\hat{\tau}_{K1} + \hat{\omega}_{K1}(\hat{\tau}_{V0})_0,\\
&\widehat{P}_{ijK1} = 2\int \big[\zeta_i - (\hat{u}_{iV0})_0\big]\big[\zeta_j - (\hat{u}_{jV0})_0\big]\widehat{\Phi}_{K1}\,\mathrm{d}\boldsymbol{\zeta},\\
&\widehat{Q}_{iK1} = \int \big[\zeta_i - (\hat{u}_{iV0})_0\big]\big[\zeta_j - (\hat{u}_{jV0})_0\big]^2 \widehat{\Phi}_{K1}\,\mathrm{d}\boldsymbol{\zeta} - \frac{3}{2}(\hat{p}_{V0})_0\hat{u}_{iK1} - \big(\widehat{P}_{ijV0}\big)_0\hat{u}_{jK1}.
\end{aligned} \tag{85}$$

The half-space problem, equation (76) with the boundary conditions (78) and (82), of the linearized Boltzmann equation is, by the following transformation of variables, seen to be the combination of two well-known Knudsen-layer problems: shear flow problem (e.g., Ohwada et al. [23] for a hard-sphere gas) and temperature-jump problem (e.g., Sone et al. [24] for a hard-sphere gas):

$$x \longleftrightarrow (\hat{\omega}_{V0})_0\eta, \quad \zeta_i \longleftrightarrow \frac{\zeta_i - \hat{u}_{wi}}{\hat{\tau}_w^{1/2}} \quad \text{with } \zeta_1 \longleftrightarrow \frac{\zeta_i n_i}{\hat{\tau}_w^{1/2}},$$

$$\phi E \longleftrightarrow \frac{\hat{\tau}_w^{3/2}}{(\hat{\omega}_{V0})_0}\widehat{\Phi}_{K1} \quad \big[E(\zeta) = \pi^{-3/2}\exp\big(-\zeta_i^2\big), \zeta = \big(\zeta_i^2\big)^{1/2}\big]. \tag{86}$$

That is, equation (76) is reduced to

$$\zeta_1 \frac{\partial \phi}{\partial x} = \mathcal{L}_0[\phi] \quad (x > 0), \tag{87}$$

where $\mathcal{L}_0[\phi]$ is the linearized collision integral $\mathcal{L}[\phi]$ defined in Appendix A (see also equation (B15)) with $\widehat{B}_{\hat{\tau}_w}$ (or $\widehat{B}_{(\hat{\tau}_{V0})_0}$) in place of $\widehat{B}_{\hat{\tau}_{V0}}$:

$$\widehat{B}_{\hat{\tau}_w}\big(\big|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})\big|, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|\big) = \hat{\tau}_w^{-1/2}\widehat{B}\big(\big|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})\big|\hat{\tau}_w^{1/2}, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|\hat{\tau}_w^{1/2}\big). \tag{88}$$

For a hard-sphere gas, $\widehat{B}_{\hat{\tau}_w}(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_*-\boldsymbol{\zeta})|, |\boldsymbol{\zeta}_*-\boldsymbol{\zeta}|)$ is independent of $\hat{\tau}_w$. For the BKW equation, $\mathcal{L}_0[\phi]$ is given by the following expression:

$$\hat{\tau}_w^{1/2}\mathcal{L}_0[\phi] = \int \left[1 + 2\zeta_i\zeta_{i*} + \frac{2}{3}\left(\zeta_i^2 - \frac{3}{2}\right)\left(\zeta_{j*}^2 - \frac{3}{2}\right)\right]\phi(\zeta_{k*})E(\zeta_*)\,\mathrm{d}\boldsymbol{\zeta}_* - \phi. \tag{89}$$

The boundary conditions are, at $x = 0$,

$$\phi = -\varpi_1 - 2\zeta_i\frac{\hat{u}_{iV1}}{\hat{\tau}_w^{1/2}} - \frac{\hat{\tau}_{V1}}{\hat{\tau}_w}\left(\zeta_i{}^2 - \frac{3}{2}\right) + \frac{\zeta_1 A(\zeta)}{\hat{\omega}_{V0}\hat{\tau}_w}\frac{\partial\hat{\tau}_{V0}}{\partial y} + \frac{\zeta_1\zeta_i B(\zeta)}{\hat{\omega}_{V0}\hat{\tau}_w^{1/2}}\frac{\partial\hat{u}_{iV0}}{\partial y} \quad (\zeta_1 > 0), \tag{90}$$

where

$$\varpi_1 = -\frac{\hat{\tau}_{V1}}{2\hat{\tau}_w} + \sqrt{\pi}\left(\frac{\hat{u}_{1V1}}{\hat{\tau}_w^{1/2}} + 2\int_{\zeta_1<0}\zeta_1\phi E\,\mathrm{d}\boldsymbol{\zeta}\right), \tag{91}$$

and, as $x \to \infty$,

$$\phi \to 0. \tag{92}$$

As will be shown with the aid of the theorem by Bardos et al. [22], the above boundary-value problem has a solution if and only if the undetermined boundary data $\hat{u}_{iV1}$ and $\hat{\tau}_{V1}$ take special values, and the solution as well as the set of the values is unique. Their theorem also assures that the solution decays with an exponential speed as $\eta \to \infty$. The boundary data $\hat{u}_{iV1}$ and $\hat{\tau}_{V1}$ are specified in the following form: at $x_i = x_{wi}$,

$$\hat{u}_{iV1}t_i^{(\alpha)} = -\frac{k_0}{\hat{\omega}_{V0}}\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}, \qquad \hat{u}_{iV1}n_i = 0, \qquad \hat{\tau}_{V1} = \frac{d_1}{\hat{\omega}_{V0}}\frac{\partial\hat{\tau}_{V0}}{\partial y}, \tag{93}$$

where k_0 and d_1 are functions of $\hat{\tau}_w$ determined by the molecular model. Their accurate values are obtained for a hard sphere gas and the BKW model:

$$k_0 = -1.2540, \qquad d_1 = 2.4001 \text{ (hard-sphere)}, \tag{94a}$$

$$k_0 = -1.01619\hat{\tau}_w^{1/2}, \qquad d_1 = 1.30272\hat{\tau}_w^{1/2} \text{ (BKW)}. \tag{94b}$$

(The numerical data for the BKW are taken from Sone and Onishi [25].)

The macroscopic variables $\hat{\omega}_K$, $\hat{u}_{iK}$, $\hat{\tau}_K$, $\hat{p}_K$, $\widehat{P}_{ijK}$, and $\widehat{Q}_{iK}$ are expressed as follows:

$$\hat{\omega}_{K1} = \frac{1}{\hat{\tau}_w}\left(\frac{\partial\hat{\tau}_{V0}}{\partial y}\right)_0\Omega_1(\tilde{\eta}), \qquad \hat{\tau}_{K1} = \frac{1}{(\hat{\omega}_{V0})_0}\left(\frac{\partial\hat{\tau}_{V0}}{\partial y}\right)_0\Theta_1(\tilde{\eta}), \tag{95a}$$

$$\hat{u}_{iK1}t_i^{(\alpha)} = -\frac{1}{(\hat{\omega}_{V0})_0}\left(\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}\right)_0 Y_0(\tilde{\eta}), \qquad \hat{u}_{iK1}n_i = 0, \tag{95b}$$

$$\widehat{P}_{ijK1}n_in_j = \widehat{P}_{ijK1}n_it_j^{(\alpha)} = \widehat{P}_{ijK1}t_i^{(\alpha)}t_j^{(3-\alpha)} = 0, \tag{95c}$$

$$\widehat{P}_{ijK1}t_i^{(\alpha)}t_j^{(\alpha)} = \frac{3}{2}\left(\frac{\partial\hat{\tau}_{V0}}{\partial y}\right)_0\left[\Omega_1(\tilde{\eta}) + \Theta_1(\tilde{\eta})\right], \quad \tilde{\eta} = \frac{(\hat{\omega}_{V0})_0}{g_M(\hat{\tau}_w)}\eta, \tag{95d}$$

$$\widehat{Q}_{iK1}n_i = 0, \qquad \widehat{Q}_{iK1}t_i^{(\alpha)} = \left(\hat{\tau}_{V0}\frac{\partial\hat{u}_{iV0}t_i^{(\alpha)}}{\partial y}\right)_0 H_A(\tilde{\eta}), \tag{95e}$$

where Ω_1, Y_0, Θ_1, and H_A are functions of $\tilde{\eta}$ and $\hat{\tau}_w$, and their functional forms depend on molecular models. The function $g_M(\hat{\tau}_w)$ of $\hat{\tau}_w$ is introduced, since the dependence of the above functions on $\hat{\tau}_w$ is expressed in a simpler form for some molecular model including a hard-sphere gas and the BKW model. That is, if the function $\widehat{B}_{\hat{\tau}_w}$ (or the linearized collision operator $\mathcal{L}_0[*]$) multiplied by $g_M(\hat{\tau}_w)$ is independent of $\hat{\tau}_w$, as for a hard-sphere gas ($g_M(\hat{\tau}_w)=1$) and for the BKW model ($g_M(\hat{\tau}_{V0})=\hat{\tau}_{V0}^{1/2}$), then the functions $\Omega_1(\tilde{\eta})/g_M(\hat{\tau}_w)$, $Y_0(\tilde{\eta})/g_M(\hat{\tau}_w)$, $\Theta_1(\tilde{\eta})/g_M(\hat{\tau}_w)$, and $H_A(\tilde{\eta})/g_M(\hat{\tau}_w)$ as well as $k_0/g_M(\hat{\tau}_w)$ and $d_1/g_M(\hat{\tau}_w)$ are independent of $\hat{\tau}_w$. These functions, denoted by $\Omega_1(\eta)$, $Y_0(\eta)$, $\Theta_1(\eta)$, and $H_A(\eta)$ with $\eta=\tilde{\eta}$ (here) in Sone [12], are tabulated for a hard-sphere gas and the BKW model there. If $\widehat{B}_{\hat{\tau}_w}$ or $\mathcal{L}_0[*]$ is not of the above-mentioned form, the choice of $g_M(\hat{\tau}_w)$ is arbitrary, but the functions $\Omega_1(\tilde{\eta})$, $Y_0(\tilde{\eta})$, etc. naturally depend on the form of $g_M(\hat{\tau}_w)$ as well as $\hat{\tau}_w$.

The slip in the boundary condition for the viscous boundary-layer equations and the Knudsen-layer correction appear at the order of ε (or $\sqrt{\mathrm{Kn}}$). This is because the length scale of variation of the viscous boundary layer in the direction normal to the boundary is of the order of εL, e.g., $n_i\partial\hat{\tau}_V/\partial x_i=\mathrm{O}(\hat{\tau}_V/\varepsilon)$. Accordingly, the thermal creep flow (Kennard [26], Sone [27], Ohwada et al. [23]) does not appear at this order, since it is proportional to the mean free path and the tangential derivative of the boundary temperature.

In the following analysis, the boundary condition for the normal velocity $\hat{u}_{iV2}n_i$ at $y=0$ is required, for which the information of $\hat{u}_{iK2}n_i$ is also necessary. From equation (84), this is expressed by the velocity distribution function $\widehat{\Phi}_{K2}$ as

$$(\hat{\omega}_{V0})_0\hat{u}_{iK2}n_i=\int \zeta_i n_i\widehat{\Phi}_{K2}\,\mathrm{d}\boldsymbol{\zeta}, \tag{96}$$

which is simplified with the aid of the second relation in each of equations (93) and (95b). Then, from equation (77),

$$\begin{aligned}(\hat{\omega}_{V0})_0\hat{u}_{iK2}n_i=\int_\infty^\eta\bigg(\int\big[&2\widehat{J}\big((\widehat{\Phi}_{V0})_0,\widehat{\Phi}_{K2}\big)+2\widehat{J}\big((\widehat{\Phi}_{V1})_0+\eta(\partial\widehat{\Phi}_{V0}/\partial y)_0,\widehat{\Phi}_{K1}\big)\\&+\widehat{J}\big(\widehat{\Phi}_{K1},\widehat{\Phi}_{K1}\big)\big]\,\mathrm{d}\boldsymbol{\zeta}\bigg)\,\mathrm{d}\eta=0.\end{aligned} \tag{97}$$

The last relation is obtained because the integration of the collision integral vanishes. On the other hand, on a boundary with diffuse reflection (generally, on a solid boundary through which mass flux is absent), $\hat{u}_i n_i=0$. Thus, at $y=0$,

$$\hat{u}_{iV2}n_i=-\hat{u}_{iK2}n_i=0. \tag{98}$$

That is, the displacement effect of the Knudsen layer vanishes (or is of higher order). Then from equation (64), $\hat{\omega}_{V0}\hat{u}_{iV2}n_i$ can be expressed by the lower-order quantities as

$$\begin{aligned}\hat{\omega}_{V0}\hat{u}_{iV2}n_i=&-\hat{\omega}_{V1}\hat{u}_{iV1}n_i-\sum_{\alpha=1}^{2}\chi_{\alpha,\alpha}\int_0^y\frac{\partial(\hat{\omega}_{V0}\hat{u}_{iV1}+\hat{\omega}_{V1}\hat{u}_{iV0})t_i^{(\alpha)}}{\partial\chi_\alpha}\,\mathrm{d}y\\&+2\bar{\kappa}\int_0^y\hat{\omega}_{V0}\hat{u}_{iV1}n_i\,\mathrm{d}y+\sum_{\alpha=1}^{2}(-1)^\alpha g_{3-\alpha}\int_0^y\big(\hat{\omega}_{V0}\hat{u}_{iV1}t_i^{(\alpha)}+\hat{\omega}_{V1}\hat{u}_{iV0}t_i^{(\alpha)}\big)\,\mathrm{d}y\\&-\sum_{\alpha=1}^{2}\int_0^y y\bigg(\kappa_\alpha\chi_{\alpha,\alpha}\frac{\partial\hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(\alpha)}}{\partial\chi_\alpha}+\vartheta\chi_{\alpha,\alpha}\frac{\partial\hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(3-\alpha)}}{\partial\chi_\alpha}\bigg)\,\mathrm{d}y\\&+\sum_{\alpha=1}^{2}(-1)^\alpha(\kappa_{3-\alpha}g_{3-\alpha}+\vartheta g_\alpha)\int_0^y y\hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(\alpha)}\,\mathrm{d}y.\end{aligned} \tag{99}$$

Now we will discuss the derivation of the relation (93) with the aid of Bardos' theorem. They considered the existence and uniqueness of the half-space boundary-value problem of the linearized Boltzmann equation (87) for hard sphere molecules under the following boundary conditions: at $x = 0$,

$$\phi = \left(c_0 + c_2\zeta_2 + c_3\zeta_3 + c_4\zeta_i^2\right) + f(\zeta_i) \quad (\zeta_1 > 0), \tag{100}$$

where $c_0,\ c_2,\ c_3$, and c_4 are constants and $f(\zeta_i)$ is a given function, and as $x \to \infty$,

$$\phi \to 0. \tag{101}$$

They proved that the solution of the half-space problem exists if and only if the constants $c_0,\ c_2,\ c_3$, and c_4 take a special set of values, and that the solution as well as the constants is unique. Further, the solution is shown to decay with an exponential speed as $x \to \infty$. This proposition is also true, if an inhomogeneous term that decays fast enough as $x \to \infty$ is added to equation (87).

In the half-space boundary-value problem (equations (87), (90)–(92)), putting aside the condition (91), according to Bardos' theorem, we find that the boundary values of $\varpi_1,\ \hat{u}_{iV1}t_i^{(\alpha)}/\hat{\tau}_w^{1/2}$, and $\hat{\tau}_{V1}/\hat{\tau}_w$ are expressed as linear combinations of those of $\hat{u}_{jV1}n_j/\hat{\tau}_w^{1/2}$, $(\partial\hat{\tau}_{V0}/\partial y)/\hat{\omega}_{V0}\hat{\tau}_w$, and $(\partial\hat{u}_{iV0}/\partial y)/\hat{\omega}_{V0}\hat{\tau}_w^{1/2}$. Equation (91), however, shows that ϖ_1 is given by another linear combination of the quantities, including ϕ, that are determined, according to the above discussion, by linear combinations of the boundary values of $\hat{u}_{jV1}n_j/\hat{\tau}_w^{1/2}$, $(\partial\hat{\tau}_{V0}/\partial y)/\hat{\omega}_{V0}\hat{\tau}_w$, and $(\partial\hat{u}_{iV0}/\partial y)/\hat{\omega}_{V0}\hat{\tau}_w^{1/2}$. Eliminating ϖ_1 from the two independent equations for it, then we obtain a relation among the boundary values of $\hat{u}_{jV1}n_j/\hat{\tau}_w^{1/2}$, $(\partial\hat{\tau}_{V0}/\partial y)/\hat{\omega}_{V0}\hat{\tau}_w$, and $(\partial\hat{u}_{iV0}/\partial y)/\hat{\omega}_{V0}\hat{\tau}_w^{1/2}$. Thus, finally we obtain the relation (93).

The analysis has been carried out under the diffuse reflection condition. It is simple to generalize the condition. The difference is only in Knudsen-layer analysis. The non-slip condition (49) is derived for a very general condition. (See Sone and Aoki [11] for the explicit condition.) The slip condition (93) and the Knudsen-layer corrections (95a)–(95e) for a hard-sphere gas under the Maxwell-type boundary condition remain in the same form with difference of numerical data of the slip coefficients k_0 and d_1 and the Knudsen-layer functions Ω_1, Y_0, etc., which are tabulated in Ohwada and Sone [20] and Wakabayashi et al. [28]. For the mathematical discussion for the Maxwell-type boundary condition, we need a more general theorem given by Coron et al. [29].

3.5. Connection of Hilbert and viscous boundary-layer solutions

In the study of a flow with its Mach number of the order of unity around a solid boundary in the preceding subsections, we found possible types of solutions: the solution (Hilbert solution) in the general domain and that (viscous boundary-layer solution + Knudsen-layer solution) in the neighbourhood (within the distance of the order of the square root of the Knudsen number) of the boundary, but have not finished the connection of these solutions. Here, we will do that. For this purpose, we will review the Hilbert and viscous boundary-layer solutions.

The velocity distribution function of the Hilbert solution is expressed as a given function of the molecular velocity with the macroscopic variables, density, flow velocity, and temperature, as the parametric functions. Up to the order of ε, it is Maxwellian, and the variation of the parametric macroscopic variables is determined by the Euler set of equations. In the neighbourhood εL of a boundary, the parametric functions are expressed in the following form:

$$\hat{h}_H = (\hat{h}_{h0})_0 + \varepsilon\left[(\hat{h}_{h1})_0 + yn_i\left(\frac{\partial\hat{h}_{h0}}{\partial x_i}\right)_0\right] + \cdots, \tag{102}$$

where the order with respect to ε is reshuffled.

In section 3.4, it was shown that the solution $\widehat{\Phi}$ in this neighbourhood is given as

$$\widehat{\Phi} = \widehat{\Phi}_V + \widehat{\Phi}_K. \tag{103}$$

If $\widehat{\Phi}_V$ is expressed as the sum:

$$\widehat{\Phi}_V = \widehat{\Phi}_H + \widehat{\Phi}_{VC}, \tag{104}$$

where $\widehat{\Phi}_{VC}$ is a correction function decaying rapidly (or faster than any inverse power of y) as $y \to \infty$, then the connection of this solution and the Hilbert solution can be made. We will derive the connection conditions of the two solutions.

The viscous solution $\widehat{\Phi}_V$ is also expressed by a given function of molecular velocity with the macroscopic variables as the parametric functions. The first three terms of the expansion in ε are given by equations (46), (55) and (63), the first two of which are repeated here for easier understanding of the process of analysis:

$$\widehat{\Phi}_{V0} = \frac{\hat{\omega}_{V0}}{(\pi\hat{\tau}_{V0})^{3/2}} \exp(-\tilde{\zeta}^2), \quad \tilde{\zeta} = \sqrt{\tilde{\zeta}_i^2}, \quad \tilde{\zeta}_i = \frac{(\zeta_i - \hat{u}_{iV0})}{\hat{\tau}_{V0}^{1/2}}, \tag{105}$$

$$\widehat{\Phi}_{V1} = \widehat{\Phi}_{V0}\left[\frac{\hat{\omega}_{V1}}{\hat{\omega}_{V0}} + 2\tilde{\zeta}_i\frac{\hat{u}_{iV1}}{\hat{\tau}_{V0}^{1/2}} + \frac{\hat{\tau}_{V1}}{\hat{\tau}_{V0}}\left(\tilde{\zeta}_i^2 - \frac{3}{2}\right) - \frac{\tilde{\zeta}_i n_i A(\tilde{\zeta})}{\hat{\omega}_{V0}\hat{\tau}_{V0}}\frac{\partial\hat{\tau}_{V0}}{\partial y} - \frac{\tilde{\zeta}_i\tilde{\zeta}_j n_j B(\tilde{\zeta})}{\hat{\omega}_{V0}\hat{\tau}_{V0}^{1/2}}\frac{\partial\hat{u}_{iV0}}{\partial y}\right]. \tag{106}$$

The function $\widehat{\Phi}_{V0}$ is Maxwellian. This is a favourable feature since $\widehat{\Phi}_{h0}$ is also Maxwellian. If we consider a point where y is large, but εy is small, then the correction $\widehat{\Phi}_{VC}$ is negligible there, since its exponential decay is assumed. If the $\widehat{\Phi}_{V0}$ agrees with $\widehat{\Phi}_{h0}$ there, $\widehat{\Phi}_{V0}$ is connected with $\widehat{\Phi}_{h0}$. (The reshuffled form of $\widehat{\Phi}_{hm}$, or equation (102), in the thin layer should be used here.) This agreement is satisfied if the following conditions are fulfilled: As $y \to \infty$,

$$\hat{\omega}_{V0} \sim (\hat{\omega}_{h0})_0, \qquad \hat{u}_{iV0} \sim (\hat{u}_{ih0})_0, \qquad \hat{\tau}_{V0} \sim (\hat{\tau}_{h0})_0. \tag{107}$$

The next-order viscous boundary-layer solution $\widehat{\Phi}_{V1}$ is not Maxwellian, but it reduces to it in the region where y is large but εy is small. In fact, $\partial\hat{\tau}_{V0}/\partial y$ and $\partial\hat{u}_{iV0}/\partial y$ in the non-Maxwellian terms of $\widehat{\Phi}_{V1}$ are small quantities of the order of ε, since the rapidly varying part (or the terms of $\partial/\partial y = \mathrm{O}(1)$) is negligibly small there owing to its exponential decay and therefore only the moderately varying part (or the terms of $\partial/\partial y = \mathrm{O}(\varepsilon)$) remains there. Thus, $\widehat{\Phi}_{V1}$ reshuffled as well as $\widehat{\Phi}_{h1}$ is Maxwellian there. Therefore, by the same reason as the first order term, the connection can be made if the macroscopic variables satisfy the following conditions: As $y \to \infty$,

$$\hat{\omega}_{V1} \sim (\hat{\omega}_{h1})_0 + yn_j\left(\frac{\partial\hat{\omega}_{h0}}{\partial x_j}\right)_0, \qquad \hat{u}_{iV1} \sim (\hat{u}_{ih1})_0 + yn_j\left(\frac{\partial\hat{u}_{ih0}}{\partial x_j}\right)_0, \qquad \hat{\tau}_{V1} \sim (\hat{\tau}_{h1})_0 + yn_j\left(\frac{\partial\hat{\tau}_{h0}}{\partial x_j}\right)_0. \tag{108}$$

Thus, if we can find the solutions $\hat{\omega}_{hm}$, $\hat{u}_{ihm}$, $\hat{\tau}_{hm}$ $(m = 0, 1)$ of the Euler type sets of equations and those $\hat{\omega}_{Vm}$, $\hat{u}_{iVm}$, $\hat{\tau}_{Vm}$ $(m = 0, 1)$ of the viscous boundary-layer type sets that satisfy the connection conditions (107) and (108), and confirm that the viscous boundary-layer solution approaches the state at infinity compatible with the connection conditions (107) and (108) with an exponential speed, then the connection process is finished up to the order of ε. In the next subsection, we will show how the solution satisfying the connection conditions is constructed.

3.6. Recipe for construction of the solution

We have seen in section 3.3 that from equation (51) and the boundary condition (49), $\hat{u}_{iV0}n_i$ vanishes, i.e. equation (53) holds, and that from equation (52), $\hat{p}_{V0}$ is independent of y, i.e. equation (54) holds. Then, from the connection condition (107),

$$(\hat{u}_{ih0})_0 n_i = 0. \tag{109}$$

This is used as the boundary condition for the Euler set (26)–(28) on a solid boundary. This is the standard boundary condition for the Euler set of equations in the classical gas dynamics. Solving the Euler set (26)–(28) under the boundary condition (109) (and the other necessary condition, e.g., the condition at infinity), we obtain the leading macroscopic variables $\hat{\omega}_{h0}$, $\hat{u}_{ih0}$, $\hat{\tau}_{h0}$, and $\hat{p}_{h0}$, thus $(\hat{\omega}_{h0})_0$, $(\hat{u}_{ih0})_0 t_i^{(\alpha)}$, $(\hat{\tau}_{h0})_0$, and $(\hat{p}_{h0})_0$. Then, from the connection condition (107) with the aid of equation $(14\text{d})_V$, the behaviour of $\hat{\omega}_{V0}$, $\hat{u}_{iV0}t_i^{(\alpha)}$, $\hat{\tau}_{V0}$, and $\hat{p}_{V0}$ as $y \to \infty$ is determined. From equation (54) and the condition on $\hat{p}_{V0}$ as $y \to \infty$ just obtained, $\hat{p}_{V0}$ is determined:

$$\hat{p}_{V0} = (\hat{p}_{h0})_0. \tag{110}$$

The conditions on $\hat{u}_{iV0}t_i^{(\alpha)}$ and $\hat{\tau}_{V0}$ as $y \to \infty$ just obtained, together with the conditions on $\hat{u}_{iV0}t_i^{(\alpha)}$ and $\hat{\tau}_{V0}$ at $y = 0$ given in equation (49), form the boundary conditions for viscous boundary-layer equations (57) and (59). In equations (57) and (59), $\hat{p}_{V0}$ is given by equation (110), $\hat{\omega}_{V0} = \hat{p}_{V0}/\hat{\tau}_{V0}$ by equation $(14\text{d})_V$, and a higher order quantity $\hat{\omega}_{V0}\hat{u}_{jV1}n_j$ is expressed by lower order quantities by equation (56) and the boundary condition (93) as follows:

$$\hat{\omega}_{V0}\hat{u}_{iV1}n_i = -\sum_{\alpha=1}^{2}\int_0^y \left(\chi_{\alpha,\alpha}\frac{\partial \hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(\alpha)}}{\partial \chi_\alpha} - (-1)^\alpha g_{3-\alpha}\hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(\alpha)}\right)\mathrm{d}y. \tag{111}$$

Thus, equations (57) and (59) with the supplementary equations $(14\text{d})_V$, (110), and (111) are reduced to the equations for $\hat{u}_{iV0}t_i^{(\alpha)}$ and $\hat{\tau}_{V0}$. These equations with the above-mentioned boundary conditions at infinity and at $y = 0$ determine $\hat{u}_{iV0}t_i^{(\alpha)}$ and $\hat{\tau}_{V0}$, thus all the macroscopic variables at the leading order of the expansion.

Now we proceed to determination of the quantities of the order ε. From the first in equation (107), the second in equation (108), and the expression (111) of $\hat{u}_{iV1}n_i$, including the condition of rapid approach, the value of $\hat{u}_{ih1}n_i$ on the boundary is given as follows:

$$(\hat{u}_{ih1})_0 n_i = -\int_0^\infty \left[\frac{1}{(\hat{\omega}_{h0})_0}\sum_{\alpha=1}^{2}\left(\chi_{\alpha,\alpha}\frac{\partial \hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(\alpha)}}{\partial \chi_\alpha} - (-1)^\alpha g_{3-\alpha}\hat{\omega}_{V0}\hat{u}_{iV0}t_i^{(\alpha)}\right) + \left(\frac{\partial \hat{u}_{ih0}}{\partial x_j}\right)_0 n_i n_j\right]\mathrm{d}y. \tag{112}$$

This is the boundary condition for the linearized Euler set (29)–(31); in equations (30) and (31) the coefficients of the derivatives of $\hat{u}_{ih1}$ and $\hat{\tau}_{h1}$ normal to the boundary vanish on the boundary owing to equation (109). Equation (112) is an appropriate boundary condition for the set. (Note: (a) A simpler example of this type of boundary-value problem for the linearized Euler equations is found in the analysis of a uniform flow of an inviscid gas past a thin body. (b) If $(\hat{u}_{ih0})_0 n_i > 0$, specification of the normal component $\hat{u}_{ih1}n_i$ only is not sufficient. We will give a simple explanatory example of this difference in Appendix C.) From the linearized Euler set and the boundary condition, the first-order quantities $\hat{\omega}_{h1}$, $\hat{u}_{ih1}$, $\hat{\tau}_{h1}$, and $\hat{p}_{h1}$ (thus $(\hat{\omega}_{h1})_0$, $(\hat{u}_{ih1})_0 t_i^{(\alpha)}$, $(\hat{\tau}_{h1})_0$, and $(\hat{p}_{h1})_0$) are obtained. Then, from the connection condition (108), with the aid of equation (15d)$_V$, the behaviour of $\hat{\omega}_{V1}$, $\hat{u}_{iV1}t_i^{(\alpha)}$, $\hat{\tau}_{V1}$, and $\hat{p}_{V1}$ as $y \to \infty$ is determined. These conditions give the boundary conditions for the viscous boundary-layer set of equations (58), (65) and (67) as $y \to \infty$. Their boundary conditions at $y = 0$ are given by the first and last relations in equation (93). The higher-order normal velocity $\hat{u}_{jV2}n_j$ in equations (65) and (67) is replaced by equation (99). Equations (58), (65) and (67) with the supplementary equations (15d)$_V$ and (99) and the boundary conditions at $y = 0$ and at infinity determine $\hat{\omega}_{V1}$, $\hat{u}_{iV1}t_i^{(\alpha)}$, $\hat{p}_{V1}$, and $\hat{\tau}_{V1}$, thus all the macroscopic variables at the order of ε.

4. Discussions

We have obtained the system of equations and boundary conditions that describes a flow of a slightly rarefied gas with a finite Mach number up to the order of the square root of the Knudsen number and described the features of the system and the procedure for finding the solution of the system. The viscous boundary-layer equations in the three-dimensional form obtained, however, are complicated and their character is not transparent. Thus, we here list the equations and boundary conditions in the two-dimensional case and give supplementary explanations. In the following discussion, the quantities are assumed to be uniform along χ_2 coordinate, and χ_1, which is orthogonal to χ_2 and is taken in such a way that $\chi_{1,1} = 1$, is simply denoted by χ. For simplicity, $\hat{u}_{iVm}t_i^{(1)}$ and $\hat{u}_{iVm}n_i$ are denoted, respectively, by u_m and v_m. Generally, the order of expansion is denoted by the subscript of each quantity, and the subscripts V, α, and β and the superscripts (α) and (β) are omitted (e.g., $t_i^{(1)} \to t_i$, $\kappa_1 \to \kappa$, $\kappa_2 \to 0$, $g_\alpha \to 0$, $\vartheta_\alpha \to 0$). The viscous boundary-layer equations and their boundary conditions obtained in sections 3.3–3.5 are rewritten as follows. The leading-order equations are

$$v_0 = 0, \qquad \hat{p}_0 = \hat{p}_0(\chi) = (\hat{p}_{h0})_0, \tag{113}$$

$$\hat{\omega}_0\left(u_0\frac{\partial u_0}{\partial \chi} + v_1\frac{\partial u_0}{\partial y}\right) = -\frac{1}{2}\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\chi} + \frac{1}{2}\frac{\partial}{\partial y}\left(\gamma_1\hat{\tau}_0^{1/2}\frac{\partial u_0}{\partial y}\right), \tag{114}$$

$$\frac{3}{2}\hat{\omega}_0\left(u_0\frac{\partial \hat{\tau}_0}{\partial \chi} + v_1\frac{\partial \hat{\tau}_0}{\partial y}\right) = -\hat{p}_0\left(\frac{\partial u_0}{\partial \chi} + \frac{\partial v_1}{\partial y}\right) + \gamma_1\hat{\tau}_0^{1/2}\left(\frac{\partial u_0}{\partial y}\right)^2 + \frac{5}{4}\frac{\partial}{\partial y}\left(\gamma_2\hat{\tau}_0^{1/2}\frac{\partial \hat{\tau}_0}{\partial y}\right), \tag{115}$$

where

$$\hat{\omega}_0 = \hat{p}_0/\hat{\tau}_0, \text{ and } v_1 = -\frac{1}{\hat{\omega}_0}\int_0^y \frac{\partial \hat{\omega}_0 u_0}{\partial \chi}\,\mathrm{d}y. \tag{116}$$

Their boundary conditions are

$$u_0 = \hat{u}_{wi}t_i, \qquad \hat{\tau}_0 = \hat{\tau}_w, \quad \text{at } y = 0, \tag{117}$$

$$u_0 \sim (\hat{u}_{ih0})_0 t_i, \qquad \hat{\tau}_0 \sim (\hat{\tau}_{h0})_0, \quad \text{as } y \to \infty. \tag{118}$$

The $(\hat{p}_{h0})_0$ in equation (113) and $(\hat{u}_{ih0})_0 t_i$ and $(\hat{\tau}_{h0})_0$ in equation (118) are given by the solution of the Euler set (26)–(28) under the boundary condition $(\hat{u}_{ih0})_0 n_i = v_0(y \to \infty) = 0$.

The next-order equations are

$$\hat{\omega}_0 u_0\frac{\partial u_1}{\partial \chi} + (\hat{\omega}_0 u_1 + \hat{\omega}_1 u_0)\frac{\partial u_0}{\partial \chi} + \hat{\omega}_0 v_1\frac{\partial u_1}{\partial y} + (\hat{\omega}_0 v_2 + \hat{\omega}_1 v_1)\frac{\partial u_0}{\partial y} - \kappa\hat{\omega}_0 u_0 v_1 + \kappa y\hat{\omega}_0 u_0\frac{\partial u_0}{\partial \chi}$$
$$= -\frac{1}{2}\left(\frac{\partial \hat{p}_1}{\partial \chi} + \kappa y\frac{\partial \hat{p}_0}{\partial \chi}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left[\gamma_1\hat{\tau}_0^{1/2}\left(\frac{\partial u_1}{\partial y} + \kappa u_0\right) + \hat{\tau}_1\frac{\mathrm{d}\gamma_1\hat{\tau}_0^{1/2}}{\mathrm{d}\hat{\tau}_0}\frac{\partial u_0}{\partial y}\right] - \kappa\gamma_1\hat{\tau}_0^{1/2}\frac{\partial u_0}{\partial y}, \tag{119}$$

$$\frac{\partial \hat{p}_1}{\partial y} = -2\kappa\hat{\omega}_0 u_0^2, \tag{120}$$

$$\begin{aligned}
&\frac{3}{2}\hat{\omega}_0 v_1 \frac{\partial \hat{\tau}_1}{\partial y} + \frac{3}{2}(\hat{\omega}_0 v_2 + \hat{\omega}_1 v_1)\frac{\partial \hat{\tau}_0}{\partial y} + \frac{3}{2}\hat{\omega}_0 u_0 \frac{\partial \hat{\tau}_1}{\partial \chi} + \frac{3}{2}(\hat{\omega}_0 u_1 + \hat{\omega}_1 u_0 + \kappa y \hat{\omega}_0 u_0)\frac{\partial \hat{\tau}_0}{\partial \chi} \\
&\quad = -\hat{p}_0\left[\frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial \chi} + \kappa\left(y\frac{\partial u_0}{\partial \chi} - v_1\right)\right] - \hat{p}_1\left(\frac{\partial v_1}{\partial y} + \frac{\partial u_0}{\partial \chi}\right) \\
&\qquad + \left[2\gamma_1\hat{\tau}_0^{1/2}\left(\frac{\partial u_1}{\partial y} + \kappa u_0\right) + \hat{\tau}_1\frac{\mathrm{d}\gamma_1\hat{\tau}_0^{1/2}}{\mathrm{d}\hat{\tau}_0}\frac{\partial u_0}{\partial y}\right]\frac{\partial u_0}{\partial y} \\
&\qquad + \frac{5}{4}\frac{\partial}{\partial y}\left[\gamma_2\hat{\tau}_0^{1/2}\frac{\partial \hat{\tau}_1}{\partial y} + \hat{\tau}_1\frac{\mathrm{d}\gamma_2\hat{\tau}_0^{1/2}}{\mathrm{d}\hat{\tau}_0}\frac{\partial \hat{\tau}_0}{\partial y}\right] - \frac{5}{4}\kappa\gamma_2\hat{\tau}_0^{1/2}\frac{\partial \hat{\tau}_0}{\partial y},
\end{aligned} \tag{121}$$

where $\hat{\omega}_1$ and v_2 are given by

$$\begin{aligned}
&\hat{\omega}_1\hat{\tau}_0 = \hat{p}_1 - \hat{\omega}_0\hat{\tau}_1, \\
&v_2 = -\frac{\hat{\omega}_1 v_1}{\hat{\omega}_0} - \frac{1}{\hat{\omega}_0}\int_0^y \left(\frac{\partial(\hat{\omega}_0 u_1 + \hat{\omega}_1 u_0)}{\partial \chi} - \kappa\hat{\omega}_0 v_1 + \kappa y\frac{\partial \hat{\omega}_0 u_0}{\partial \chi}\right)\mathrm{d}y.
\end{aligned} \tag{122}$$

The boundary conditions at $y = 0$ are

$$u_1 = -\frac{k_0}{\hat{\omega}_0}\frac{\partial u_0}{\partial y}, \qquad \hat{\tau}_1 = \frac{d_1}{\hat{\omega}_0}\frac{\partial \hat{\tau}_0}{\partial y}. \tag{123}$$

The conditions at infinity are as follows: as $y \to \infty$,

$$\hat{\omega}_1 \sim (\hat{\omega}_{h1})_0 + yn_j\left(\frac{\partial \hat{\omega}_{h0}}{\partial x_j}\right)_0, \qquad u_1 \sim (\hat{u}_{ih1})_0 t_i + yn_j\left(\frac{\partial \hat{u}_{ih0}}{\partial x_j}\right)_0 t_i, \qquad \hat{\tau}_1 \sim (\hat{\tau}_{h1})_0 + yn_j\left(\frac{\partial \hat{\tau}_{h0}}{\partial x_j}\right)_0.$$

The data on the right-hand sides are given by the solution of the Euler set of equations (29)–(31) under the boundary condition:

$$(\hat{u}_{ih1})_0 n_i = -\int_0^\infty \left[\left(\frac{\partial \hat{u}_{ih0}}{\partial x_j}\right)_0 n_i n_j + \frac{1}{(\hat{\omega}_{h0})_0}\frac{\partial \hat{\omega}_0 u_0}{\partial \chi}\right]\mathrm{d}y. \tag{124}$$

The viscous boundary-layer equations and boundary conditions (114)–(118) at the leading order are the same as those for the Navier–Stokes equations for a compressible gas (the compressible gas version of the Prandtl boundary-layer equations and their boundary conditions). The next order equations also do not contain the non-Navier–Stokes stress and heat flow terms. The term containing $\hat{\tau}_1\,\mathrm{d}\gamma_1\hat{\tau}_0^{1/2}/\mathrm{d}\hat{\tau}_0$ or $\hat{\tau}_1\,\mathrm{d}\gamma_2\hat{\tau}_0^{1/2}/\mathrm{d}\hat{\tau}_0$ is due to the fact that the viscosity or thermal conductivity of the gas depends on its temperature. These equations are derived from the Navier–Stokes set of equations for a compressible gas where the coordinate normal to the boundary is stretched by the factor of the square root $\sqrt{\mathrm{Re}}$ of the Reynolds number, by a power series expansion in $\mathrm{Re}^{-1/2}$. (Note the relation (10), which depends on the stress formula (68) in converting the mean free path to the viscosity, for the transfer from the $\mathrm{Re}^{-1/2}$-expansion to ε-expansion.) The result does not support the claim by Darrozes [6] that the boundary-layer equations describing the leading effect of gas rarefaction should contain a non-Navier–Stokes stress term. (In his paper no explicit equations are given.) Owing to special anisotropic character of the viscous boundary-layer equations, a higher-order quantity, or v_2, which is expressed by lower-order quantities, enters the equations that determine the behaviour of $\hat{\omega}_1$, u_1, $\hat{p}_1$, and $\hat{\tau}_1$. However, in view of the fact that its boundary value $(v_2)_0$ vanishes owing to the displacement effect of the Knudsen layer being of higher order, the contributions up to the order of $\sqrt{\mathrm{Kn}}$ are included in the system of the Navier–Stokes equations (in the nonexpanded original form) and the (correspondingly rearranged) slip conditions consisting of tangential velocity slip due to the shear of flow and temperature jump due to the temperature gradient normal to the boundary.

In the analysis of the viscous boundary layer, we do not follow the method of our previous analyses of this type of problem, that is, we do not split the solution into two parts: the Hilbert solution and its correction. This is because the viscous boundary-layer equation is usually obtained in the form without splitting and we want to keep the resemblance here. We can, of course, carry out the analysis in the split form: $\widehat{\Phi}_V = \widehat{\Phi}_H + \widehat{\Phi}_{VC}$, where $\widehat{\Phi}_{VC}$ is the viscous boundary-layer correction to the Hilbert solution and is assumed to decay rapidly away from the boundary. Correspondingly, a macroscopic variable $\hat{h}_V$ is split as

$$\hat{h}_{V0} = (\hat{h}_{h0})_0 + \hat{h}_{VC0}, \qquad \hat{h}_{V1} = (\hat{h}_{h1})_0 + y n_i (\partial \hat{h}_{h0}/\partial x_i)_0 + \hat{h}_{VC1}, \quad \ldots,$$

where $\hat{h}$ represents $\hat{\omega}$, $\hat{u}_i$, etc. The analysis of $\widehat{\Phi}_{VC}$ is carried out in a similar way to that of $\widehat{\Phi}_V$. The equations that determine the behaviour of $\hat{h}_{VC0}$, $\hat{h}_{VC1}$, etc. are naturally those obtained by substituting the above relations into the equations for $\hat{h}_{V0}$, $\hat{h}_{V1}$, etc. An equation for $\hat{h}_{VCm}$ is denoted by the superscript * on the number of its corresponding equation. From equation (53)* and the condition of rapid decay of $\hat{h}_{VCm}$, it is found that $(\hat{u}_{ih0})_0 n_i = 0$, which is the boundary condition (109) for the Euler set (26)–(28). From its solution, $(\hat{\omega}_{h0})_0$, $(\hat{u}_{ih0})_0 t_i^{(\alpha)}$, $(\hat{\tau}_{h0})_0$, and $(\hat{p}_{h0})_0$ are determined. From equation (54)* and the condition of rapid decay of $\hat{p}_{VC0}$, it is found that $\hat{p}_{VC0} = 0$. Equations (57)* and (59)* with the supplementary equations (111)* and $(14\mathrm{d})_V^*$ are the equations for the correction functions $\hat{\omega}_{VC0}$, $\hat{u}_{iVC0} t_i^{(\alpha)}$, $\hat{u}_{iVC1} n_i$, and $\hat{\tau}_{VC0}$; equation (49)* and the condition of rapid vanishing of $\hat{h}_{VC0}$ as $y \to \infty$ are their boundary conditions. This system of equations and boundary conditions contains an undetermined boundary value $(\hat{u}_{ih1})_0 n_i$ as well as the determined data $(\hat{\omega}_{h0})_0$, $(\hat{u}_{ih0})_0 t_i^{(\alpha)}$, $(\partial \hat{u}_{ih0}/\partial x_j)_0 n_i n_j$, $(\hat{\tau}_{h0})_0$, and $(\hat{p}_{h0})_0$. The undetermined data $(\hat{u}_{ih1})_0 n_i$ should be a special value for the system to have a solution. That is, taking the limit as $y \to \infty$ of equation (111)* with $y(\hat{\omega}_{h0} \partial \hat{u}_{ih0}/\partial x_j)_0 n_i n_j$ term shifted to the right-hand side and taking into account the condition of rapid vanishing, then we obtain equation (112)*. (The condition does not generally assure the existence of solution.) This is the boundary condition for the Euler set (29)–(31). The discussion for the further steps can be carried out in a similar way. In this procedure, the connection conditions are replaced by the condition of rapid decay, and the discussion may be more comprehensible. The two ways of analysis are equivalent.

We have not mentioned about the shock layer (Grad [30]). A thin layer with sharp variation across it may appear inside the gas region when a local Mach number in the flow is larger than unity. It is not difficult to incorporate the layer in the present solution if the layer is outside the viscous boundary layer (or in the Euler region). Up to the first order of ε, in the same way as in the classical gas dynamics, the layer may be taken as a discontinuity, and the regions across the discontinuity have only to be connected by the shock wave condition (Rankine–Hugoniot relation). The internal structure of the layer is expressed by that of the plane shock wave (Caflisch and Nicolaenko [31]).

In the present analysis, we considered a flow of a finite Mach number and obtained the Euler or viscous boundary-layer equations for a compressible gas as the fluid-dynamic type equations. The analysis for a flow of a small Mach number but a large Reynolds number can be carried out with small modification, and the Euler set of equations and the set of viscous boundary-layer equations for an incompressible gas are derived as the leading-order system. The situation where the Mach number Ma is of the order of $\mathrm{Kn}^{n/m}$ (m and n $(< m)$: positive integers) and the variation of temperature of the boundary is the same order as the Mach number is considered. Then, the leading Maxwellian distribution $\widehat{\Phi}_{h0}$ or $\widehat{\Phi}_{V0}$ is a uniform state at rest (or $\hat{\omega}_{h0}$ and $\hat{\tau}_{h0}$ are constant and $\hat{u}_{ih0} = 0$, or $\hat{\omega}_{V0}$ and $\hat{\tau}_{V0}$ are constant and $\hat{u}_{iV0} = 0$), and a nonuniform term starts at the order of

$\mathrm{Kn}^{n/m}$. The Reynolds number Re ($\propto$ Ma/Kn) is of the order of $\mathrm{Kn}^{-(m-n)/m}$. Thus, the thickness of the viscous boundary layer may be considered to be of the order of $\mathrm{Kn}^{(m-n)/2m}L$; correspondingly, the stretching factor of the viscous boundary-layer coordinate is $\mathrm{Kn}^{-(m-n)/2m}$. With these differences in mind and taking $\mathrm{Kn}^{1/2m}$ as the expansion parameter instead of ε (or $\mathrm{Kn}^{1/2}$), the analysis (the Euler region, the viscous boundary layer, the Knudsen layer, and their connection) can be carried out in a similar way to that in section 3.

5. Concluding remarks

We have developed an asymptotic theory of a rarefied gas flow of a finite Mach number in a general domain for small Knudsen numbers. The behaviour of the gas is conveniently described by splitting the domain into three regions: Euler region (or the overall region except in the neighbourhood of a boundary), viscous boundary layer, and Knudsen layer. The solution is expressed in a power series of the square root of the Knudsen number, and in the former two regions, the behaviour of the gas is determined by equations of fluid-dynamic type (or equations governing the component functions of the expansion of the macroscopic variables, density, flow velocity, and temperature, of the gas): equations of Euler type in the Euler region and those of viscous boundary-layer type in the viscous boundary layer. Up to the order of $\sqrt{\mathrm{Kn}}$, these equations are those obtained by the expansion of the Euler set of equations in the Euler region, and the equations obtained by the expansion of the compressible Navier–Stokes set of equations in the viscous boundary layer. (Note: The relation Re $= (2/\gamma_1)(10/3\pi)^{1/2}$Ma/Kn between Kn and Re, which is the result of the relation between the mean free path and the viscosity given by the nondimensional form of the stress formula (68), should be taken into account in the expansion, where γ_1 is evaluated at $\hat{\tau}_{V0} = 1$ for a general molecular model.) At the order of $\sqrt{\mathrm{Kn}}$, the viscous boundary-layer equations should be solved under the slip conditions (or the first and last relations in equation (93)) for the tangential velocity and temperature. In these equations, the component function at the order Kn of the expansion of the flow velocity normal to the boundary, which is expressed by the lower-order quantities, is contained owing to strong anisotropy of the viscous boundary layer. Because the boundary value of this component vanishes ($(\hat{u}_{iV2}n_i)_0 = 0$, equation (98)) owing to the displacement effect of the Knudsen layer being of higher order, the contributions up to the order $\sqrt{\mathrm{Kn}}$ are included in the system of the non-expanded original Navier–Stokes equations and the correspondingly rearranged slip conditions (consisting of tangential velocity slip due to the shear of flow and temperature jump due to the temperature gradient normal to the boundary). The result of the present asymptotic analysis is different from that of Darrozes [6].

Finally, it is noted that the kinetic boundary condition on a body or bodies is used in deriving the supplementary equation (111) or (99) for the boundary-layer equations (57) and (59) or (58), (65), and (67), as well as equation (53). The information of the Knudsen layer is contained in the corresponding equations of higher order than those given in the present paper.

Appendix A. Solutions $\widehat{\Phi}_{V1}$ and $\widehat{\Phi}_{V2}$

The particular solution of the integral equation for $\widehat{\Phi}_{Vm}$ (equations (43), (44), etc.) is expressed by a polynomial of $\tilde{\zeta}_i$, with coefficients of functions of $\tilde{\zeta}$. This is derived from the isotropic property of the collision operator with respect to $\tilde{\zeta}_i$ as follows. (See, e.g., Sone and Aoki [11].)

Consider the following linear integral equation which is equivalent to the integral equations (43), (44), etc. for $\widehat{\Phi}_{Vm}$:

$$\mathcal{L}[\phi] = H(\zeta_i). \tag{A1}$$

Here, $H(\zeta_i)$ is a given function of ζ_i, and $\mathcal{L}[\phi]$ is the linearized collision integral defined by

$$\mathcal{L}[\phi] = \int E(\zeta_*)(\phi' + \phi'_* - \phi - \phi_*)\widehat{B}_{\hat{\tau}_{V0}}\big(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|\big)\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\zeta}_*, \tag{A2}$$

$$E(\zeta) = \pi^{-3/2}\exp(-\zeta^2), \quad \zeta = (\zeta_i^2)^{1/2},$$

where $\widehat{B}_{\hat{\tau}_{V0}}(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|)$ is

$$\widehat{B}_{\hat{\tau}_{V0}}\big(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|\big) = \hat{\tau}_{V0}^{-1/2}\widehat{B}\big(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|\hat{\tau}_{V0}^{1/2}, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|\hat{\tau}_{V0}^{1/2}\big). \tag{A3}$$

For a hard-sphere gas, $\widehat{B}_{\hat{\tau}_{V0}}(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|)$ is independent of $\hat{\tau}_{V0}$. The transformation from equation (43) or (44) to equation (A1) is carried out in the following way: change the variables ζ_i to $\tilde{\zeta}_i$ [$= (\zeta_i - \hat{u}_{iV0})/\hat{\tau}_{V0}^{1/2}$]; eliminate $\sim$ over ζ_i, etc.; make correspondence $\widehat{\Phi}_{Vm}$ to $E\phi$. For the BKW equation, the linearized collision integral $\mathcal{L}[\phi]$ is given by

$$\hat{\tau}_{V0}^{1/2}\mathcal{L}[\phi] = \int \left[1 + 2\zeta_i\zeta_{i*} + \frac{2}{3}\left(\zeta_i^2 - \frac{3}{2}\right)\left(\zeta_{j*}^2 - \frac{3}{2}\right)\right]\phi(\zeta_{k*})E(\zeta_*)\,\mathrm{d}\boldsymbol{\zeta}_* - \phi. \tag{A4}$$

We will derive the general form of the solution of the linear integral equation (A1) when $H(\zeta_i)$ has a certain symmetry as well as satisfies the solvability conditions:

$$\int (1,\ \zeta_i, \zeta_j^2)H(\zeta_k)E(\zeta)\,\mathrm{d}\boldsymbol{\zeta} = 0. \tag{A5}$$

Now take a group of integral equations:

$$\mathcal{L}[\phi_{i_1,\ldots,i_m}(\zeta_k)] = H_{i_1,\ldots,i_m}(\zeta_k), \tag{A6}$$

where the inhomogeneous terms $H_{i_1,\ldots,i_m}(\zeta_k)$ are assumed to be symmetric with respect to their subscripts $(i_1,\ldots,i_m)$ and to satisfy the following isotropic property:

$$H_{i_1,\ldots,i_m}(\ell_{kh}\zeta_h) = \ell_{i_1j_1}\cdots\ell_{i_mj_m}H_{j_1,\ldots,j_m}(\zeta_k), \quad \text{with } \ell_{ik}\ell_{jk} = \delta_{ij}. \tag{A7}$$

It is noted here that the collision integral also has this property. That is, let

$$F_{i_1,\ldots,i_m}(\zeta_k) = \mathcal{L}\big[\zeta_{i_1}\cdots\zeta_{i_m}f(\zeta)\big], \tag{A8}$$

and then

$$F_{i_1,\ldots,i_m}(\ell_{kh}\zeta_h) = \ell_{i_1j_1}\cdots\ell_{i_mj_m}F_{j_1,\ldots,j_m}(\zeta_k). \tag{A9}$$

The functions $F_{i_1,\ldots,i_m}(\zeta_k)$ are, of course, symmetric with respect to the subscripts $(i_1,\ldots,i_m)$.

The solutions $\phi_{i_1,\ldots,i_m}(\zeta_k)$ that are orthogonal to 1, ζ_i, and ζ_j^2 (the solutions of the associated homogeneous equation):

$$\int (1, \zeta_i, \zeta_j^2)\phi_{i_1,\ldots,i_m}(\zeta_k)E(\zeta)\,\mathrm{d}\boldsymbol{\zeta} = 0, \tag{A10}$$

have the same isotropic property:

$$\phi_{i_1,\ldots,i_m}(\ell_{kh}\zeta_h) = \ell_{i_1j_1}\cdots\ell_{i_mj_m}\phi_{j_1,\ldots,j_m}(\zeta_k). \tag{A11}$$

The solutions $\phi_{i_1,\ldots,i_m}(\zeta_k)$ are also symmetric with respect to the subscripts $(i_1,\ldots,i_m)$.

On the other hand, any group of functions $\Phi_{i_1,\ldots,i_m}(\zeta_k)$ that are symmetric with respect to their subscripts $(i_1,\ldots,i_m)$ and that have the isotropic property:

$$\Phi_{i_1,\ldots,i_m}(\ell_{kh}\zeta_h) = \ell_{i_1 j_1}\cdots\ell_{i_m j_m}\Phi_{j_1,\ldots,j_m}(\zeta_k), \tag{A12}$$

are known to be expressed in the following form. (See, e.g., Sone and Aoki [11] for elementary derivation.)

$$\Phi_{i_1,\ldots,i_m}(\zeta_h) = \sum_{n=0}^{[m/2]} g_n(\zeta)\sum_{*}\underbrace{\zeta_{i_a}\cdots\zeta_{i_b}}_{m-2n}\underbrace{\delta_{i_c i_d}\cdots\delta_{i_e i_f}}_{n}, \tag{A13}$$

where $g_n(\zeta)$ is an arbitrary function of ζ, the symbol $[m/2]$ is the largest integer that does not exceed $m/2$, and the summation $\sum_*$ is carried out over $m!/2^n n!(m-2n)!$ terms in the following way: divide m subscripts $(i_1,\ldots,i_m)$ into two sets: one with $m-2n$ elements $i_a,\ldots,i_b$ and the other with $2n$ elements $i_c,\ldots,i_f$, for which there are $m!/(2n)!(m-2n)!$ ways of division. For each set, consider all possible ways to make n pairs $(i_c,i_d),\ldots,(i_e,i_f)$, which are $(2n)!/2^n n!$. Then there are $m!/(2n)!(m-2n)! \times (2n)!/2^n n!$ terms. The $\sum_*$ means to sum up all these terms. For some small values of m, $\Phi_{i_1,\ldots,i_m}(\zeta_h)$ is given as follows:

$$\Phi(\zeta_h) = g_0(\zeta) \quad (m=0), \qquad \Phi_i(\zeta_h) = \zeta_i g_0(\zeta) \quad (m=1),$$
$$\Phi_{i,j}(\zeta_h) = \zeta_i\zeta_j g_0(\zeta) + \delta_{ij}g_1(\zeta) \quad (m=2),$$
$$\Phi_{i,j,k}(\zeta_h) = \zeta_i\zeta_j\zeta_k g_0(\zeta) + (\zeta_i\delta_{jk} + \zeta_j\delta_{ki} + \zeta_k\delta_{ij})g_1(\zeta) \quad (m=3).$$

With this form of function as $\phi_{i_1,\ldots,i_m}(\zeta_k)$, solution of the integral equation (A1) for a function of three independent variables $\zeta_1,$ $\zeta_2,$ and ζ_3 is reduced to that of an integral equation for a function or functions of one independent variable ζ. For example, when $H(\zeta_i)$ in equation (A1) is given by $\zeta_1(\zeta^2-5/2)$, then ϕ is expressed as $\zeta_1 g(\zeta)$, where $g(\zeta)$ is the solution of the integral equation:

$$\mathcal{L}\bigl[\zeta_1 g(\zeta)\bigr] = \zeta_1(\zeta^2-5/2), \quad \text{with the orthogonal conditon } \int_0^\infty \zeta^4 g(\zeta)E(\zeta)\,\mathrm{d}\zeta = 0.$$

Arranging the inhomogeneous terms in equations (43) and (44) in such a way that the above discussion of solution of the integral equation (A6) may be applied, we find that the solutions $\widehat{\Phi}_{Vm}$ $(m=1,2)$ are expressed in the form given in equations (55) and (63) with the aid of the functions $A(\tilde{\zeta})$, $B(\tilde{\zeta})$, etc. defined in Appendix B. In this process of arrangement of the inhomogeneous term, $\widehat{J}$ terms are treated separately, since each of any subdivision of the $\widehat{J}$ terms satisfies the solvability conditions owing to the symmetry relation of the collision integral (see, e.g., Grad [15], Cercignani [14], Sone and Aoki [11]) but is not symmetric with respect to some of its subscripts. (See Appendix B.)

Appendix B. Functions $A(\tilde{\zeta})$, $B(\tilde{\zeta})$, etc. and γ_1, γ_2, etc.

The functions $A(\tilde{\zeta})$, $B(\tilde{\zeta})$, etc., appeared in section 3.3, are expressed by linear combinations of solutions of the following integral equations with a subsidiary condition (indicated by sc) related to equation (A10). In this Appendix the symbols ζ_i and ζ are used instead of $\tilde{\zeta}_i$ and $\tilde{\zeta}$ for simplicity.

$$\mathcal{L}\bigl[\zeta_i A(\zeta)\bigr] = -\zeta_i\left(\zeta^2 - \frac{5}{2}\right), \quad \text{sc: } \int_0^\infty \zeta^4 A(\zeta)E(\zeta)\,\mathrm{d}\zeta = 0; \tag{B1}$$

$$\mathcal{L}\left[\left(\zeta_i\zeta_j - \frac{1}{3}\zeta^2\delta_{ij}\right)\mathcal{B}^{(m)}(\zeta)\right] = IB_{ij}^{(m)}; \tag{B2}$$

$$\mathcal{L}\big[(\zeta_i\delta_{jk} + \zeta_j\delta_{ki} + \zeta_k\delta_{ij})\mathcal{T}_1^{(m)}(\zeta) + \zeta_i\zeta_j\zeta_k\mathcal{T}_2^{(m)}(\zeta)\big] = IT_{ijk}^{(m)},$$
$$\text{sc: } \int_0^\infty \big(5\zeta^4\mathcal{T}_1^{(m)} + \zeta^6\mathcal{T}_2^{(m)}\big)E(\zeta)\,\mathrm{d}\zeta = 0; \tag{B3}$$

$$\mathcal{L}\big[\zeta_i\delta_{jk}\widetilde{\mathcal{T}}_{11}^{(0)}(\zeta) + (\zeta_j\delta_{ki} + \zeta_k\delta_{ij})\widetilde{\mathcal{T}}_{12}^{(0)}(\zeta) + \zeta_i\zeta_j\zeta_k\widetilde{\mathcal{T}}_2^{(0)}(\zeta)\big] = I\widetilde{T}_{i,jk}^{(0)},$$
$$\text{sc: } \int_0^\infty \big(5\zeta^4\widetilde{\mathcal{T}}_{12}^{(0)} + \zeta^6\widetilde{\mathcal{T}}_2^{(0)}\big)E(\zeta)\,\mathrm{d}\zeta = 0, \quad \int_0^\infty \big(5\zeta^4\widetilde{\mathcal{T}}_{11}^{(0)} + \zeta^6\widetilde{\mathcal{T}}_2^{(0)}\big)E(\zeta)\,\mathrm{d}\zeta = 0; \tag{B4}$$

$$\mathcal{L}\big[(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathcal{Q}_1^{(0)}(\zeta)$$
$$+ (\zeta_i\zeta_j\delta_{kl} + \zeta_i\zeta_k\delta_{jl} + \zeta_i\zeta_l\delta_{jk} + \zeta_j\zeta_k\delta_{il} + \zeta_j\zeta_l\delta_{ik} + \zeta_k\zeta_l\delta_{ij})\mathcal{Q}_2^{(0)}(\zeta) + \zeta_i\zeta_j\zeta_k\zeta_l\mathcal{Q}_3^{(0)}(\zeta)\big] = IQ_{ijkl}^{(0)},$$
$$\text{sc: } \int_0^\infty (1, \zeta^2)\big(15\zeta^2\mathcal{Q}_1^{(0)} + 10\zeta^4\mathcal{Q}_2^{(0)} + \zeta^6\mathcal{Q}_3^{(0)}\big)E(\zeta)\,\mathrm{d}\zeta = 0; \tag{B5}$$

$$\mathcal{L}\big[\delta_{ij}\delta_{kl}\widetilde{\mathcal{Q}}_{11}^{(0)}(\zeta) + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\widetilde{\mathcal{Q}}_{12}^{(0)}(\zeta) + (\zeta_i\zeta_j\delta_{kl} + \zeta_k\zeta_l\delta_{ij})\widetilde{\mathcal{Q}}_{21}^{(0)}(\zeta)$$
$$+ (\zeta_i\zeta_k\delta_{jl} + \zeta_i\zeta_l\delta_{jk} + \zeta_j\zeta_k\delta_{il} + \zeta_j\zeta_l\delta_{ik})\widetilde{\mathcal{Q}}_{22}^{(0)}(\zeta) + \zeta_i\zeta_j\zeta_k\zeta_l\widetilde{\mathcal{Q}}_3^{(0)}(\zeta)\big] = I\widetilde{Q}_{ij,kl}^{(0)},$$
$$\text{sc: } \int_0^\infty (1, \zeta^2)\big(15\zeta^2\widetilde{\mathcal{Q}}_{11}^{(0)} + 10\zeta^4\widetilde{\mathcal{Q}}_{21}^{(0)} + \zeta^6\widetilde{\mathcal{Q}}_3^{(0)}\big)E(\zeta)\,\mathrm{d}\zeta = 0,$$
$$\text{and } \int_0^\infty (1, \zeta^2)\big(15\zeta^2\widetilde{\mathcal{Q}}_{12}^{(0)} + 10\zeta^4\widetilde{\mathcal{Q}}_{22}^{(0)} + \zeta^6\widetilde{\mathcal{Q}}_3^{(0)}\big)E(\zeta)\,\mathrm{d}\zeta = 0; \tag{B6}$$

$$\mathcal{L}\big[\mathcal{N}^{(m)}(\zeta)\big] = IN^{(m)}, \quad \text{sc: } \int_0^\infty (1,\ \zeta^2)\zeta^2\mathcal{N}^{(m)}E(\zeta)\,\mathrm{d}\zeta = 0. \tag{B7}$$

Here, $\mathcal{L}[\phi]$ is the linearized collision integral defined by equation (A2), where $\widehat{B}_{\hat{\tau}_{V0}}$ is given by equation (A3). For the BKW equation, it is given by equation (A4). The inhomogeneous terms $IB_{ij}^{(m)}$, $IT_{ijk}^{(m)}$, etc. in equations (B2)–(B7) are as follows:

$$IB_{ij}^{(0)} = -2\left(\zeta_i\zeta_j - \frac{\zeta^2}{3}\delta_{ij}\right), \qquad IB_{ij}^{(1)} = \left(\zeta_i\zeta_j - \frac{\zeta^2}{3}\delta_{ij}\right)A(\zeta),$$
$$IB_{ij}^{(2)} = \left(\zeta_i\zeta_j - \frac{\zeta^2}{3}\delta_{ij}\right)\left(2(\zeta^2 - 3)A(\zeta) - \zeta\frac{\partial A(\zeta)}{\partial\zeta} + 2\hat{\tau}_{V0}\frac{\partial A(\zeta)}{\partial\hat{\tau}_{V0}}\right),$$
$$IB_{ij}^{(3)} = \frac{1}{E(\zeta)}\left(J_{\hat{\tau}_{V0}}\big(\zeta_iA(\zeta)E(\zeta), \zeta_jA(\zeta)E(\zeta)\big) - \frac{\delta_{ij}}{3}\sum_{k=1}^{3}J_{\hat{\tau}_{V0}}\big(\zeta_kA(\zeta)E(\zeta), \zeta_kA(\zeta)E(\zeta)\big)\right),$$
$$IB_{ij}^{(4)} = \left(\zeta_i\zeta_j - \frac{\zeta^2}{3}\delta_{ij}\right)\mathcal{B}^{(0)}(\zeta); \tag{B8}$$

$$IT_{ijk}^{(0)} = -\zeta_i\zeta_j\zeta_k\mathcal{B}^{(0)}(\zeta) + \gamma_1(\zeta_i\delta_{jk} + \zeta_j\delta_{ki} + \zeta_k\delta_{ij}),$$
$$IT_{ijk}^{(1)} = -\zeta_i\zeta_j\zeta_k\left(2A(\zeta) - \frac{1}{\zeta}\frac{\partial A(\zeta)}{\partial\zeta}\right),$$

$$IT_{ijk}^{(2)} = -\zeta_i\zeta_j\zeta_k\left((\zeta^2-3)\mathcal{B}^{(0)}(\zeta) - \frac{\zeta}{2}\frac{\partial\mathcal{B}^{(0)}(\zeta)}{\partial\zeta}\right) + \frac{\gamma_1}{2}(\zeta_i\delta_{jk}+\zeta_j\delta_{ki}+\zeta_k\delta_{ij}),$$

$$IT_{ijk}^{(3)} = -\zeta_i\zeta_j\zeta_k\hat{\tau}_{V0}\frac{\partial\mathcal{B}^{(0)}(\zeta)}{\partial\hat{\tau}_{V0}} + \hat{\tau}_{V0}\frac{\mathrm{d}\gamma_1}{\mathrm{d}\hat{\tau}_{V0}}(\zeta_i\delta_{jk}+\zeta_j\delta_{ki}+\zeta_k\delta_{ij}); \tag{B9}$$

$$I\widetilde{T}_{i,jk}^{(0)} = J_{\hat{\tau}_{V0}}\big(\zeta_i A(\zeta)E(\zeta),\, \zeta_j\zeta_k\mathcal{B}^{(0)}(\zeta)E(\zeta)\big)/E(\zeta); \tag{B10}$$

$$IQ_{ijkl}^{(0)} = \frac{1}{3}\left[\zeta^2\mathcal{B}^{(0)}(\zeta) + 2\gamma_1\left(\zeta^2-\frac{3}{2}\right)\right](\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$
$$-\zeta_i\zeta_j\zeta_k\zeta_l\left(2\mathcal{B}^{(0)}(\zeta) - \frac{1}{\zeta}\frac{\partial\mathcal{B}^{(0)}(\zeta)}{\partial\zeta}\right); \tag{B11}$$

$$I\widetilde{Q}_{ij,kl}^{(0)} = J_{\hat{\tau}_{V0}}\big(\zeta_i\zeta_j\mathcal{B}^{(0)}(\zeta)E(\zeta),\, \zeta_k\zeta_l\mathcal{B}^{(0)}(\zeta)E(\zeta)\big)/E(\zeta); \tag{B12}$$

$$IN^{(0)} = 2\zeta^2\left(\zeta^2-\frac{7}{2}\right)A(\zeta) - \zeta^3\frac{\partial A(\zeta)}{\partial\zeta},$$
$$IN^{(1)} = 2\hat{\tau}_{V0}\frac{\partial A(\zeta)}{\partial\hat{\tau}_{V0}}\zeta^2 - 5\hat{\tau}_{V0}\frac{\mathrm{d}\gamma_2}{\mathrm{d}\hat{\tau}_{V0}}\left(\zeta^2-\frac{3}{2}\right),$$
$$IN^{(2)} = \zeta^2 A(\zeta) - \frac{5}{2}\gamma_2\left(\zeta^2-\frac{3}{2}\right),$$
$$IN^{(3)} = \frac{1}{E(\zeta)}\sum_{k=1}^{3}J_{\hat{\tau}_{V0}}\big(\zeta_k A(\zeta)E(\zeta),\, \zeta_k A(\zeta)E(\zeta)\big). \tag{B13}$$

The summation sign $\sum$ is used in $IB_{ij}^{(3)}$ and $IN^{(3)}$, since the summation convention is a little difficult to be distinguished there, and the comma in the subscripts on the inhomogeneous terms $I\widetilde{T}_{i,jk}^{(0)}$ and $I\widetilde{Q}_{ij,kl}^{(0)}$ are put to show their non-symmetric property with respect to the subscripts. The $J_{\hat{\tau}_{V0}}(\phi,\psi)$, γ_1, and γ_2 in some of the inhomogeneous terms in equations (B2)–(B7) are defined as follows: The $J_{\hat{\tau}_{V0}}(\phi,\psi)$ is given by $\widehat{J}(\phi,\psi)$ (equation (2)) where $\widehat{B}$ is replaced by $\widehat{B}_{\hat{\tau}_{V0}}$ (equation (A3)), i.e.

$$J_{\hat{\tau}_{V0}}(\phi,\psi) = \frac{1}{2}\int(\phi'_*\psi' + \phi'\psi'_* - \phi_*\psi - \phi\psi_*)\widehat{B}_{\hat{\tau}_{V0}}\big(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_*-\boldsymbol{\zeta})|, |\boldsymbol{\zeta}_*-\boldsymbol{\zeta}|\big)\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\zeta}_*. \tag{B14}$$

Incidentally, the linearized collision integral $\mathcal{L}[\phi]$ defined by equation (A2) is related to $J_{\hat{\tau}_{V0}}(\phi,\psi)$:

$$E(\zeta)\mathcal{L}[\phi] = 2J_{\hat{\tau}_{V0}}(E, E\phi). \tag{B15}$$

The γ_1 and γ_2 are given by the moments of $\mathcal{B}^{(0)}(\zeta)$ and $A(\zeta)$:

$$\gamma_1 = I_6(\mathcal{B}^{(0)}), \qquad \gamma_2 = 2I_6(A), \quad I_n(Z) = \frac{8}{15\sqrt{\pi}}\int_0^\infty \zeta^n Z(\zeta)\exp(-\zeta^2)\,\mathrm{d}\zeta. \tag{B16}$$

It is noted that some terms (e.g., $J_{\hat{\tau}_{V0}}(\zeta_i E(\zeta), \zeta_j A(\zeta)E(\zeta))$) of $\widehat{J}(\widehat{\Phi}_{V1}, \widehat{\Phi}_{V1})$ in the inhomogeneous term of equation (44) can be transformed into simpler forms for which the solution is easily found. (Its detailed explanation is omitted because of limited space.) For the BKW equation, the $J_{\hat{\tau}_{V0}}$ terms in the above inhomogeneous terms are taken to be zero.

The inhomogeneous terms of the integral equations (B1)–(B13) satisfy the solvability conditions. The forms of the solutions are guaranteed by the discussion in Appendix A. (See Sone and Aoki [11] for the details.) The inhomogeneous terms $I\widetilde{T}^{(0)}_{i,jk}$ and $I\widetilde{Q}^{(0)}_{ij,kl}$ are not symmetric with respect to some of their subscripts. Thus, the statement in Appendix A is not directly applied, but the forms of the solutions corresponding to these inhomogeneous terms are verified by simple generalization of the argument in Sone and Aoki [11]. (Note: the present forms of the solutions are determined under the condition that the isotropic condition (A11) hold for both the cases of the determinant $\|\ell_{ij}\| = 1$ and -1.) The subsidiary conditions are the conditions that the solution is orthogonal to the solutions of the corresponding homogeneous equation, such as equation (A10). In equation (B2), the orthogonal conditions are incorporated in the form of the solution in view of the fact that the traces, with respect to the subscript (i, j), of the inhomogeneous term (B8) vanishes. Except for a hard-sphere gas, the solutions of the integral equations, $A(\zeta)$, $\mathcal{B}^{(m)}(\zeta)$, $\mathcal{T}^{(m)}(\zeta)$, etc., depend on $\hat{\tau}_{V0}$.

The functions $B(\zeta)$, $\mathcal{B}_1(\zeta)$, $\mathcal{Q}_1(\zeta)$, etc. as well as $A(\zeta)$ in the solutions $\widehat{\Phi}_{V1}$ and $\widehat{\Phi}_{V2}$ are expressed by linear combinations of the solutions of the above integral equations as

$$\begin{aligned}
&B(\zeta) = \mathcal{B}^{(0)}(\zeta), \qquad \mathcal{B}_1(\zeta) = -\mathcal{B}^{(1)}(\zeta), \qquad \mathcal{B}_2(\zeta) = -\mathcal{B}^{(2)}(\zeta) - 2\mathcal{B}^{(3)}(\zeta),\\
&\mathcal{N}^A(\zeta) = -\frac{1}{3}\left[\mathcal{N}^{(0)}(\zeta) + \mathcal{N}^{(1)}(\zeta) + \mathcal{N}^{(2)}(\zeta) + 2\mathcal{N}^{(3)}(\zeta)\right],\\
&\mathcal{N}^B(\zeta) = -\frac{1}{3}\mathcal{N}^{(2)}(\zeta), \qquad \mathcal{T}^A_N(\zeta) = \mathcal{T}^{(0)}_N(\zeta) \quad (N = 1, 2),\\
&\mathcal{T}^B_1(\zeta) = \mathcal{T}^{(1)}_1(\zeta) + \mathcal{T}^{(2)}_1(\zeta) + \mathcal{T}^{(3)}_1(\zeta) - 2\widetilde{\mathcal{T}}^{(0)}_{12}(\zeta),\\
&\mathcal{T}^B_2(\zeta) = \mathcal{T}^{(1)}_2(\zeta) + \mathcal{T}^{(2)}_2(\zeta) + \mathcal{T}^{(3)}_2(\zeta) - 2\widetilde{\mathcal{T}}^{(0)}_2(\zeta),\\
&\mathcal{Q}_1(\zeta) = \mathcal{Q}^{(0)}_1(\zeta) - \widetilde{\mathcal{Q}}^{(0)}_{12}(\zeta) - \frac{1}{3}\zeta^2\mathcal{B}^{(4)}(\zeta), \qquad \mathcal{Q}_2(\zeta) = \mathcal{Q}^{(0)}_2(\zeta) - \widetilde{\mathcal{Q}}^{(0)}_{22}(\zeta),\\
&\mathcal{Q}_3(\zeta) = \mathcal{Q}^{(0)}_2(\zeta) - \widetilde{\mathcal{Q}}^{(0)}_{22}(\zeta) + \mathcal{B}^{(4)}(\zeta), \qquad \mathcal{Q}_4(\zeta) = \mathcal{Q}^{(0)}_3(\zeta) - \widetilde{\mathcal{Q}}^{(0)}_3(\zeta).
\end{aligned} \tag{B17}$$

Similarly to γ_1 and γ_2, the γ_3, γ_7, γ_8, γ_9, and γ_{10}, appeared in the formulas of stress tensor and heat flow vector (equations (68) and (69)) as well as in the fluid-dynamic type equations (65)–(67), are defined by the following moments of solutions of the integral equations:

$$\begin{aligned}
&\gamma_3 = 2I_6(\mathcal{B}_1) = 5I_6(\mathcal{T}^A_1) + I_8(\mathcal{T}^A_2), \qquad \gamma_7 = I_6(\mathcal{B}_2), \qquad \gamma_8 = I_6(\mathcal{Q}_2) + \frac{1}{7}I_8(\mathcal{Q}_4),\\
&\gamma_9 = -I_6(\mathcal{Q}_3 - \mathcal{Q}_2) = -I_6(\mathcal{B}^{(4)}), \qquad \gamma_{10} = \frac{5}{8}I_6(\mathcal{T}^B_1) + \frac{1}{8}I_8(\mathcal{T}^B_2).
\end{aligned} \tag{B18}$$

The second equality in the equation for γ_3 is derived with the aid of self-adjoint property of the linearized collision operator $\mathcal{L}(\phi)$.

The numerical solutions of $A(\zeta)$, $B(\zeta)$, $\mathcal{B}^{(1)}(\zeta)$ (as $F(\zeta)$), $\mathcal{T}^{(0)}_1(\zeta)$ (as $D_1(\zeta)$), and $\mathcal{T}^{(0)}_2(\zeta)$ (as $D_2(\zeta)$) for a hard-sphere gas are given in Ohwada and Sone [20]. (The function $\nu(\zeta)$ in equations (A1)–(A5) on page 409 in the paper is a misprint for $2\sqrt{2}\nu(\zeta)$.) Some of γ_m are

$$\gamma_3 = 1.947906, \qquad \gamma_7 = 0.188106.$$

For the BKW model,

$$\begin{aligned}
&A(\zeta)/\hat{\tau}^{1/2}_{V0} = \zeta^2 - 5/2, \qquad B(\zeta)/\hat{\tau}^{1/2}_{V0} = 2, \qquad \mathcal{B}^{(1)}(\zeta)/\hat{\tau}_{V0} = -\zeta^2 + 5/2,\\
&\mathcal{T}^{(0)}_1(\zeta)/\hat{\tau}_{V0} = -1, \qquad \mathcal{T}^{(0)}_2(\zeta)/\hat{\tau}_{V0} = 2; \qquad \gamma_3 = \gamma_7 = \gamma_8 = \gamma_9 = \gamma_{10} = \hat{\tau}_{V0}.
\end{aligned}$$

(The definitions of $A(\zeta)$, $B(\zeta)$, and γ_m for the BKW model are modified from those in our previous papers, Ohwada and Sone [20] and Sone et al. [32], etc. In the present definitions, we can save separate comments on the equations in section 3.3, etc. for the BKW model, although $A(\zeta)$, $B(\zeta)$, and γ_m contain $\hat{\tau}_{V0}$ in the new definitions.)

Appendix C. On the boundary condition for the linearized Euler set of equations

Here we will give a simple explanatory example of the difference of the appropriate number of boundary conditions for the linearized Euler set of equations depending on situations. Consider a two-dimensional case of an incompressible gas (ρ is constant). The flow is assumed here to be very close to a uniform flow. The uniform flow, naturally, satisfies the Euler set of equations. Let (X, Y) be the space coordinates, and let $p_0 + \tilde{p}$ and $(U + u(X, Y), v(X, Y))$ (p_0 and U: positive constants), respectively, be the pressure and the flow velocity, where $\tilde{p}$, u, and v are small quantities.

The equations for the perturbed quantities u, v, and $\tilde{p}$ are the linearized Euler set:

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} = 0, \qquad U\frac{\partial u}{\partial X} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial X}, \qquad U\frac{\partial v}{\partial X} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial Y}. \tag{C1}$$

The coefficients of $\partial u/\partial Y$ and $\partial v/\partial Y$ vanish in the last two equations. In terms of the stream function Ψ: $u = \partial\Psi/\partial Y$, $v = -\partial\Psi/\partial X$, the set of equations (C1) is reduced to the single equation for the vorticity $\Delta\Psi$:

$$\frac{\partial}{\partial X}\Delta\Psi = 0, \tag{C2}$$

which shows the invariance of vorticity along the streamlines of the uniform flow.

(I) Consider a uniform flow past a thin body set parallel to the flow, given by $y = f^{\pm}(x)$ $(0 \leqslant x \leqslant L)$, where f^+ or f^- expresses the upper or lower surface of the body and $|f^{\pm}|$ is very small. Then the boundary condition that $v_i n_i = 0$ on the body can be replaced by the condition $v = U\,\mathrm{d}f^{\pm}/\mathrm{d}X$ at $Y = 0_{\pm}$ $(0 \leqslant X \leqslant L)$. The condition at upstream infinity is $u = v = 0$. From the condition at upstream infinity, we obtain the Laplace equation $\Delta\Psi = 0$ over the whole field. The boundary conditions are $\partial\Psi/\partial X = -U\,\mathrm{d}f^{\pm}/\mathrm{d}X$ at $Y = 0_{\pm}$ $(0 \leqslant X \leqslant L)$ and $\Psi = 0$ at infinity.

This boundary-value problem of the Laplace equation has unique solution Ψ, if the difference of the values of Ψ at the origin and at infinity is specified, which corresponds to the circulation around the body. In other words, the boundary-value problem for the linearized Euler set of equations under the boundary conditions that v is given on $Y = 0_+$ and 0_- $(0 \leqslant X \leqslant L)$ and that u, v, and $\tilde{p}$ approach zero as $X \to -\infty$ determines the flow field except the circulation around the body, which cannot be determined within the framework of the steady Euler set of equations.

The problem that we are discussing in the present paper is an extension of this example. Let the solution $\hat{\omega}_{h0}$, $\hat{u}_{ih0}$, and $\hat{\tau}_{h0}$ (or $\hat{p}_{h0}$) of the leading order Euler set of equations be given. From the linearized Euler set of equations, equations governing the variation of perturbed vorticity and entropy along the streamlines of the solution of the leading order are derived. The streamlines do not pass the body and they are along the boundary of the body, since $(\hat{u}_{ih0})_0 n_i = 0$ (equation (109)). The perturbed vorticity and entropy are determined by their upstream conditions. This is an important point. To make this point clearer, consider the following problem.

(II) Consider an artificial problem. The difference from problem (I) is the position and the property of the body. A body of length L and of zero thickness is set at $X = 0$ and between $0 \leqslant Y \leqslant L$. The body is assumed to disturb the uniform flow a little. Then from the condition $u = v = 0$ at upstream infinity $(X \to -\infty)$, we obtain the Laplace equation $\Delta\Psi = 0$ only in $(-\infty < X < \infty, \quad Y < 0$ and $Y > L)$ and in $(-\infty < X < 0, \quad 0 \leqslant Y \leqslant L)$ but $\Delta\Psi = F(Y)$ in $(0 < X < \infty, \quad 0 \leqslant Y \leqslant L)$, where $F(Y)$ is an undetermined function of Y. Then we need a condition (or an upstream condition) to determine $F(Y)$ on $X = 0_+$ $(0 \leqslant Y \leqslant L)$, where the normal component of the velocity of the uniform flow is positive, in addition to the general boundary condition for the Laplace equation. Further, it may be noted that no additional condition is required on $X = 0_-$ $(0 \leqslant Y \leqslant L)$, into which the streamlines of the unperturbed flow go.

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